Chapter 9.1

Infinite Series

Mathematically, a **sequence** is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than by the standard function notation. For instance, in the sequence

$$
\begin{array}{ccccccc}
1, & 2, & 3, & 4, & \ldots, & n, & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
a_1, & a_2, & a_3, & a_4, & \ldots, & a_n, & \ldots
\end{array}
\qquad \text{Sequence}
$$

$$\{a_n\} = \{3 + (-1)^n\}$$

$$\{b_n\} = \left\{\frac{n}{1-2n}\right\}$$

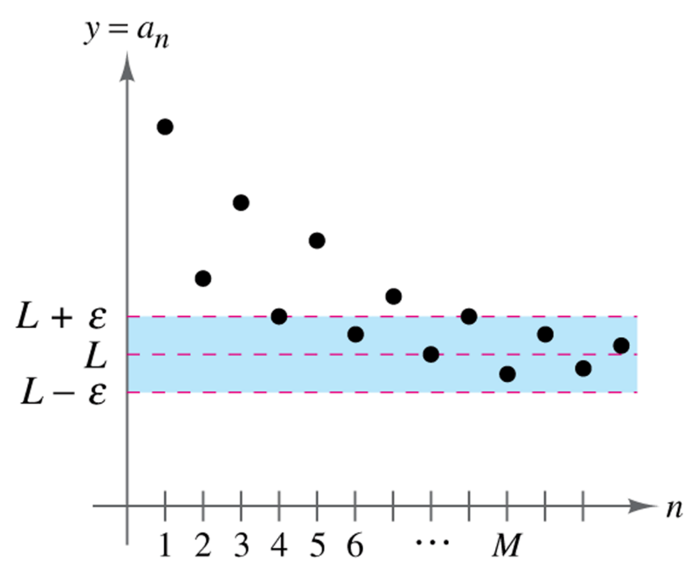
Definition of the Limit of a Sequence and Figure 9.1

Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges.**



Theorem 9.1 Limit of a Sequence

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Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

Find the limit of the sequence whose nth term is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Theorem 9.2 Properties of Limits of Sequences

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Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n\to\infty} ca_n = cL$, c is any real number
3. $\lim_{n\to\infty} (a_n b_n) = LK$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

$$\{a_n\} = \{3 + (-1)^n\}$$

$$\{b_n\} = \left\{\frac{n}{1-2n}\right\}$$

(Use Th 9.1, then L'Hôpital)

Theorem 9.3 Squeeze Theorem for Sequences

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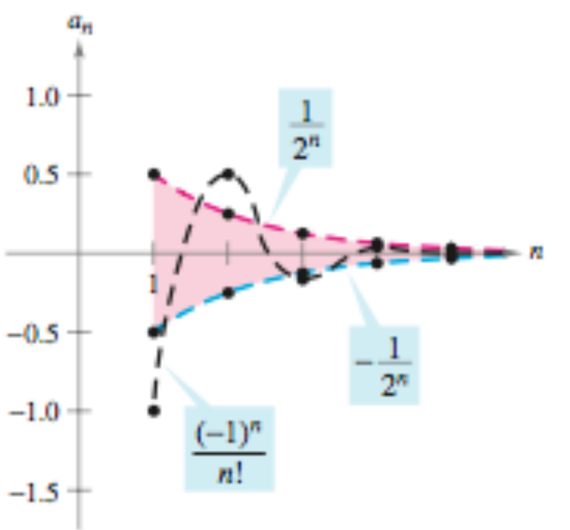
If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$



For $n \ge 4$, $(-1)^n/n!$ is squeezed between $-1/2^n$ and $1/2^n$.
Figure 9.2

$$\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\} \text{ converges!}$$

eventually (for n≥4),

$$2^n < n! \text{ so } \frac{1}{2^n} > \frac{1}{n!}$$

Theorem 9.4 Absolute Value Theorem

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For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \qquad \text{then} \qquad \lim_{n\to\infty} a_n = 0.$$

Proof Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$. Because both of these sequences converge to 0 and

$$-|a_n| \le a_n \le |a_n|$$

you can use the Squeeze Theorem to conclude that $\{a_n\}$ converges to 0.

EXAMPLE 6 Finding the nth Term of a Sequence

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine whether the particular sequence you have chosen converges or diverges.

(Use Th 9.1, then L'Hôpital)

$$\{a_n\} = \left\{(-1)^n \left(\frac{3^n - 1}{n!}\right)\right\} \text{ converges to } 0$$

(Use Th 9.4, then growth on n!>3^n)

Definition of a Monotonic Sequence

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A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdot \cdot \cdot \leq a_n \leq \cdot \cdot \cdot$$

or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdot \cdot \cdot \geq a_n \geq \cdot \cdot \cdot.$$

Determine whether each sequence having the given nth term is monotonic.

a. $a_n = 3 + (-1)^n$ **b.** $b_n = \frac{2n}{1+n}$ **c.** $c_n = \frac{n^2}{2^n - 1}$

Definition of a Bounded Sequence

Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

EXAMPLE 9 Bounded and Monotonic Sequences

a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic and so, by Theorem 9.5, must converge.

b. The divergent sequence $\{b_n\} = \{n^2/(n+1)\}$ is monotonic, but not bounded. (It is bounded below.)

c. The divergent sequence $\{c_n\} = \{(-1)^n\}$ is bounded, but not monotonic.