Improper Integrals   Name: Block: Seat:

Improper Integrals are integrals where:

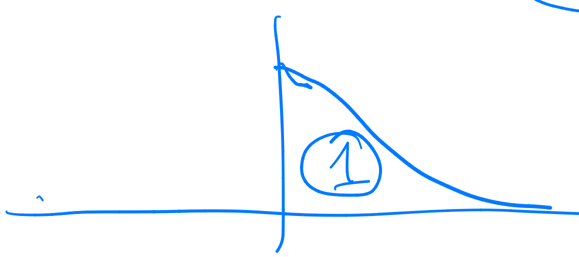
- One or both limits of integration are $\pm\infty$
  (sometimes called "Type I" or "horizontal" improper integrals)
- The integrand is undefined at one or both of the limits of integration or in between the limits of integration
  (the integrand has a vertical asymptote, sometimes called "Type II" or "vertical" improper integrals)

Improper integrals are said to be

- **convergent** if the limit is finite and that limit is the value of the improper integral.
- **divergent** if the limit does not exist.

Examples

1. $\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\frac{-1}{e^b} + e^0 = 0 + 1 = 1$



typo $+$

2. $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx = \int_0^{\infty}\frac{1}{1+x^2}\,dx$

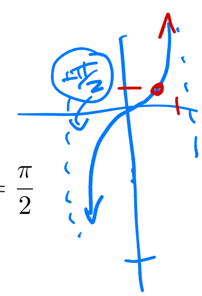
typo

First we consider $\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\left[\arctan x\right]_b^0 = \arctan 0 - \lim_{b\to-\infty}\arctan b = 0 - \frac{\pi}{2} = \frac{\pi}{2}$

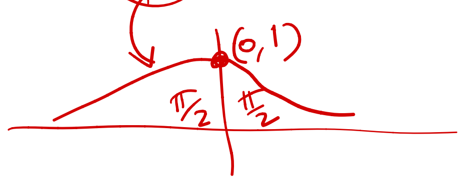
$-\infty$ typo

Similarly $\int_0^{\infty}\frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\left[\arctan x\right]_0^b = \lim_{b\to-\infty}\arctan b - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ ✓

∞



Putting it together, $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$



p-Test for Integrals Thm 8.7 (page 578):

If $p > 1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\displaystyle\frac{1}{p-1}$

If $p \le 1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ diverges

$p = 3/2 < 1$ so conv to $\dfrac{1}{\frac{3}{2}-1} = 2$ by p-Test for integrals

Examples

1. $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$

   Following usual methods:

   please give me fewer points

   $$\lim_{b\to\infty}\int_1^b \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_1^b x^{-3/2}\,dx = \lim_{b\to\infty}\left[\frac{1}{-\frac{1}{2}}x^{-1/2}\right]_1^b = \cancel{\lim_{b\to\infty}} - \left[\frac{2}{x^{1/2}}\right]_1^{\infty} = 0 - (-2) = 2$$

   But following the theorem:

   $p = \frac{3}{2} > 1$, hence $\dfrac{1}{p-1} = \dfrac{1}{\frac{3}{2}-1} = \dfrac{1}{\frac{1}{2}} = 2$

2. $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx$

   $p = 1/2 \le 1$ so diverge by p-test for Integrals

   Following usual methods:

   $$\lim_{b\to\infty}\int_1^b \frac{1}{x^{1/2}}\,dx = \lim_{b\to\infty}\left[2\sqrt{x}\right]_1^b = \lim_{b\to\infty} 2\sqrt{b} - 2 = \infty$$

   But following the theorem:

   $p = \frac{1}{2} \le 1$, hence it diverges

Exercises

1. $\int_1^\infty \frac{1}{x}\,dx$ (try it both ways!)

$p = 1 \leq 1$ so by p-Test for Integr diverges

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b$$

$$= \lim_{b\to\infty} \ln b - \ln 1 \text{ diverges}$$

2. $\int_1^\infty \frac{1}{x^2}\,dx$

$p = 2 > 1$, so by p-Test for Integs converges to $\frac{1}{2-1} = 1$

$$\lim_{b\to\infty}\int_1^b x^{-2}\,dx = \lim_{b\to\infty} \frac{x^{-1}}{-1}\Big|_1^b = \lim_{b\to\infty} \frac{-1}{b} - \frac{-1}{1} = 1$$

3. $\int_0^\infty \frac{1}{x^2+9}\,dx$ $a = 3$ $u = x$

Hint: $\frac{1}{x^2+9} = \frac{\frac{1}{9}}{\frac{x^2}{9}+1}$ reminds me of arctan (see p. ~~520~~ 519)

$$\lim_{b\to\infty}\int_0^b \frac{\frac{1}{9}}{\frac{x^2}{9}+1}\,dx = \lim_{b\to\infty} \frac{1}{9}\int_0^b \frac{1}{\frac{x^2}{9}+1}\,dx$$

$u = \frac{x}{3}$

$3\,du = \cancel{\frac{1}{3}}\,dx$

$\frac{3}{9}\int \frac{1}{u^2+1}\,du$

$\frac{1}{3}\arctan\frac{x}{3} + c$

$$= \lim_{b\to\infty} \frac{1}{3}\arctan\frac{x}{3}\Big|_0^b$$

$$= \lim_{b\to\infty} \frac{1}{3}\arctan\frac{b}{3} - 0 = \frac{1}{3}\cdot\frac{\pi}{2}$$

4. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
Hint: Use the limit as $b \to 0^+$

$$\lim_{b\to 0^+} \int_b^1 x^{-1/2}\,dx$$

$$\lim_{b\to 0^+} 2x^{1/2}\Big|_b^1 = 2\sqrt{1} - \lim_{b\to 0^+} 2\sqrt{b}$$

$$= 2 - 0$$

5. $\displaystyle\int_{-1}^1 \frac{1}{x^{2/3}}\,dx = \int_{-1}^0 x^{-2/3}dx + \int_0^1 x^{-2/3}dx$

$$\lim_{b\to 0^-} \int_{-1}^b x^{-2/3}\,dx = \lim_{b\to 0^-} 3x^{1/3}\Big|_{-1}^0 = 0-(-1)(3) = 3$$

$$\lim_{b\to 0^+} \int_b^1 x^{-2/3}\,dx = \lim_{b\to 0^+} 3x^{1/3}\Big|_b^1 = 3-0=3$$

$$\text{so } \int_{-1}^1 \frac{1}{x^{2/3}}dx = 3+3=6$$

6. $\displaystyle\int_{-2}^1 \frac{1}{x^2}\,dx = \int_{-2}^0 x^{-2}dx + \int_0^1 x^{-2}dx$

$$= \lim_{b\to 0^-} \frac{-1}{x}\Big|_{-2}^b + \lim_{b\to 0^+} \frac{-1}{x}\Big|_b^1$$

$$= \lim_{b\to 0^-} \frac{-1}{b} - \left(\frac{-1}{-2}\right) + \frac{-1}{1} - \lim_{b\to 0^+} \frac{-1}{x}$$

$$= \left(+\infty - \frac{1}{2}\right) + \left(-1 + -\infty\right)$$

diverges

7. $\int_0^1 x \ln x \, dx$ $= \lim_{b \to 0^+} \int_b^1 x \ln x \, dx = \left(\frac{1}{2} (\ln 1) - \frac{1}{4} \right) - \lim_{b \to 0^+} \left(\frac{b^2}{2} \ln b - \frac{b^2}{4} \right)$

$\int x \ln x \, dx$

$u = \ln x$	$x = dv$
$du = \frac{1}{x}$	$\frac{x^2}{2} = v$

$\frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx$

$\frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

$= \frac{-1}{4} - \lim_{b \to 0^+} \left(\frac{\ln b}{2 b^{-2}} \right) - \lim_{b \to 0^+} \left(\frac{b^2}{4} \right)$

$\left(-\frac{\infty}{\infty} \right)$ by L'H

$= -\frac{1}{4} - \lim_{b \to 0^+} \frac{\frac{1}{b}}{-4 b^{-3}} - 0$

$= \frac{-1}{4} - \lim_{b \to 0^+} \frac{b^3}{-4b}$

$= \frac{-1}{4} - \lim_{b \to 0^+} \frac{b^2}{-4}$

$= \frac{-1}{4} - 0$

8. $\int_0^\infty \cos x \, dx$

$\lim_{b \to \infty} \sin \Big|_0^b =$ diverges since $\lim_{b \to \infty} \sin b$ alternates

Answers: 1) diverges, 2) 1 3) $\frac{\pi}{6}$ 4) 2 5) 6 6) diverges 7) $-\frac{1}{4}$ 8) diverges