| AP CALCULUS BC | YouTube Live Virtual Lessons | Mr. Bryan Passwater<br>Mr. Anthony Record |
|---|---|---|
| **Topic:** 10.11 (Part I) | **Finding Taylor Polynomial Approximations of Functions** | **Date:** April 8, 2020 |

# Warm-Up

Consider the series $\sum_{n=1}^{\infty} a_n (x-2)^{n-1} = 4 - \frac{4}{3}(x-2) + \frac{4}{9}(x-2)^2 - \frac{4}{27}(x-2)^3 + \cdots$

(a) Find $\sum_{n=1}^{\infty} a_n (x-2)^{n-1}$ when $x = 1$

$$x = 1 \Rightarrow \sum_{n=1}^{\infty} a_n (x-2)^{n-1} = 4 - \frac{4}{3}(-1) + \frac{4}{9}(-1)^2 - \frac{4}{27}(-1)^3 + \cdots = 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots$$

$$\text{geometric with } a = 4, r = \frac{1}{3} \Rightarrow \sum_{n=1}^{\infty} a_n (-1)^{n-1} = \frac{4}{1-\frac{1}{3}} = \frac{12}{3-1} = 6$$

(b) Determine if $\sum_{n=1}^{\infty} a_n (x-2)^{n-1}$ is absolutely convergent, conditionally convergent, or divergent when $x = 4$.

$$x = 4 \Rightarrow \sum_{n=1}^{\infty} a_n (2)^{n-1} = 4 - \frac{4}{3}(2) + \frac{4}{9}(2)^2 - \frac{4}{27}(2)^3 + \cdots + (-1)^{n-1} 4\left(\frac{2}{3}\right)^{n-1} + \cdots$$

$$\lim_{n\to\infty}\left|(-1)^{n-1} 4\left(\frac{2}{3}\right)^{n-1}\right| = \lim_{n\to\infty}\left|4\left(\frac{2}{3}\right)^{n-1}\right| \Rightarrow \text{geometric with } r = \frac{2}{3} \Rightarrow \text{absolutely convergent}$$

(c) Differentiating the individual terms of the series creates a new series given below:

$$\sum_{n=1}^{\infty} b_n = -\frac{4}{3} + \frac{4 \cdot 2}{9}(x-2) - \frac{4 \cdot 3}{27}(x-2)^2 + \cdots (-1)^n \frac{4n}{3^n}(x-2)^{n-1} + \cdots$$

Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{b_n}{n^2}$

$$\sum_{n=1}^{\infty} \frac{b_n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{4n}{3^n}(x-2)^{n-1}$$

$$\lim_{n\to\infty} \left| \frac{4(n+1)(x-2)^n}{3^{n+1} \cdot (n+1)^2} \cdot \frac{3^n \cdot n^2}{4n(x-2)^{n-1}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)n}{3 \cdot (n+1)} \right| = \left| \frac{(x-2)}{3} \right| < 1 \Rightarrow |x-2| < 3 \Rightarrow -1 < x < 5$$

$$x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n(-1-2)^{n-1}}{n^2} \frac{4n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} 3^{n-1}}{n^2} \frac{4n}{3^n} = -\frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \text{ divergent } p = 1$$

$$x = 5 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n(5-2)^{n-1}}{n^2} \frac{4n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(3)^{n-1}}{n^2} \frac{4n}{3^n} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Rightarrow \text{ convergent by AST}$$

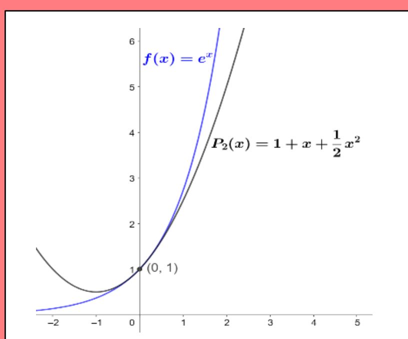
interval of convergence is $-1 < x \le 5$

## Lesson Overview

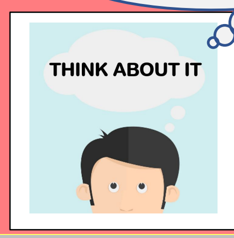
| WHAT WE ARE GOING TO DO | WHAT YOU SHOULD ALREADY KNOW |
|---|---|
| • Introduce the Taylor Polynomials and approximations<br>• Clone the world's cutest dog<br>• Begin to see a connection and application of our study of series | • Find derivatives of known functions<br>• Be familiar with polynomials<br>• Write an equation of a tangent line |

**WHAT YOU WILL BE ABLE TO DO**

Given $f(x) = sin(x)$. Find an approximation of $\sin(0.2)$ using a 2nd degree Maclaurin polynomial centered at $c = 0$.




**Question**: How do I find a second degree Taylor polynomial?




## Guided Practice

To get us thinking about how Taylor polynomials work or what they really are, let's think about something that is more understandable and awesome! Let's talk about "cloning" my dog Buddy.

Imagine we had a machine that could actually clone my dog for us.
To make this machine work, we need to program all of the properties or traits of my dog Buddy so it could make a copy.
So, let's start to feed the machine some properties ...

So, I enter the fact that Buddy is brown.

We can take a similar approach to "clone" functions. This introduces the concept of a Taylor Polynomial.

Let's address three important questions about Taylor Polynomials …

1. What exactly is a Taylor Polynomial?
2. Why are they important?
3. Have we been secretly doing these all along without realizing it?

**What is a Taylor Polynomial**?

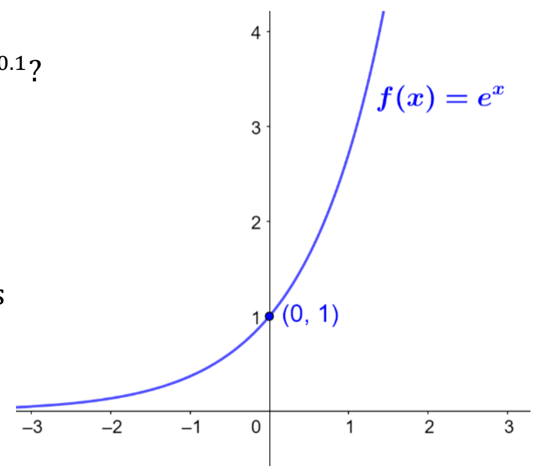
Consider the function $f(x) = e^x$

Put yourself in the year 1715 … how would you find $f(0.1) = e^{0.1}$?

We don't have a simple way to evaluate expressions like $e^{0.1}$, but could we approximate them?

This is the idea behind Taylor polynomials. Taylor polynomials are a way to approximate "ugly" functions with "prettier" ones.

Polynomials are nice functions because they only require arithmetic to evaluate




| Ugly Functions | Pretty (Nicer) Functions |
|---|---|
| $e^x$ | $1 + x + \frac{x^2}{2}$ |
| $\ln(x)$ | $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ |
| $\sin(x)$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ |
| $\cos(x)$ | $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ |
| $\tan^{-1}(x)$ | $x - \frac{x^3}{3} + \frac{x^5}{5}$ |

To create a polynomial that approximates a function, we will essentially ***clone*** the function!

**Cloning a function**

1. Pick a value $c$ where $f(x)$ and $P(x)$ have the same value:

$$f(c) = P(c)$$

We say that our polynomial is **centered at *c*** or **expanded about *c***

There are infinitely many polynomials that will pass through a given point, so we need to find more traits to match at the center.

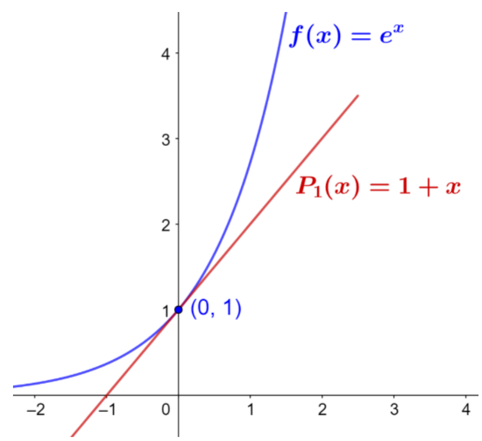
2. We can also match the slopes of our polynomial with the slope of the function:

$$f'(c) = P'(c)$$

**Example 1**: Find a first degree polynomial centered at $x = 0$ for $f(x) = e^x$

$$f(x) = e^x \qquad f'(x) = e^x$$
$$x = 0 \Rightarrow f(0) = e^0 = 1 \qquad f'(0) = e^0 = 1$$
$$P_1(x) = a_0 + a_1 x \qquad P_1'(x) = a_1$$
$$x = 0 \Rightarrow P_1(0) = a_0 \qquad P_1'(0) = a_1$$
$$P_1(0) = f(0) = 1 = a_0 \qquad P_1'(0) = f'(0) = 1 = a_1$$
$$P_1(x) = a_0 + a_1 x = 1 + x$$

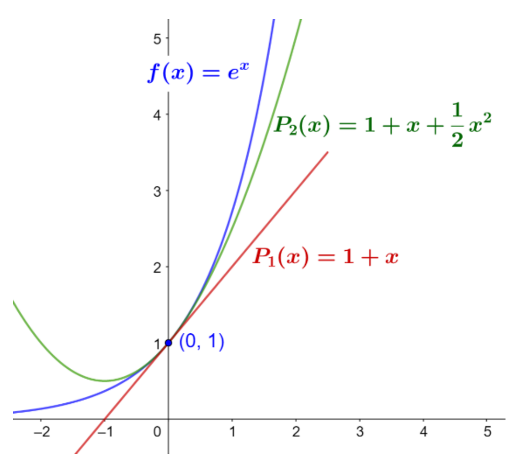
**Answer to Example 1**:




Notice that our approximation is close to $f(x)$ as long as we are close to our center. But, to get a better clone, we need to match more traits.

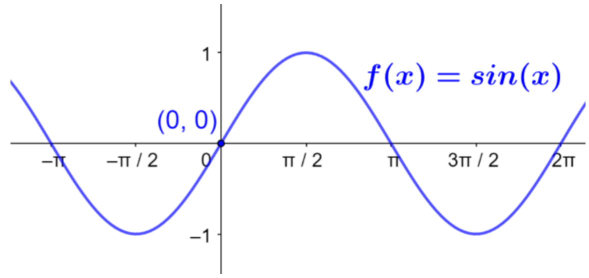
**Example 2**: Find a second degree polynomial centered at $x = 0$ for $f(x) = e^x$

$$f(x) = e^x \qquad f'(x) = e^x \qquad f''(x) = e^x$$
$$x = 0 \Rightarrow f(0) = e^0 = 1 \qquad f'(0) = e^0 = 1 \qquad f''(0) = e^0 = 1$$
$$P_2(x) = 1 + 1x + a_2 x^2 \qquad P_2'(x) = 1 + 2a_2 x \qquad P_2''(x) = 2a_2$$
$$x = 0 \Rightarrow P_2(0) = 1 \qquad P_2'(0) = 1 \qquad P_2''(0) = 2a_2$$
$$P_2''(0) = f''(0) = 1 = 2a_2 \Rightarrow a_2 = \frac{1}{2} \qquad P_2(x) = 1 + x + \frac{1}{2}x^2$$



We could continue to match up characterics to form polynomials that become better and better clones for $f(x) = e^x$. If we matched up infinitely many derivatives, we would end up creating a Power Series!

**Example 3**: Find a third degree polynomial centered at $x = 0$ for $f(x) = \sin(x)$.
Use this polynomial to approximate $\sin(0.2)$.




$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\qquad f(x) = \sin(x)$

$P_3'(x) = a_1 + 2a_2 x + 3a_3 x^2$ $\qquad f'(x) = \cos(x)$

$P_3''(x) = 2a_2 + 6a_3 x$ $\qquad f''(x) = -\sin(x)$

$P_3'''(x) = 6a_3$ $\qquad f'''(x) = -\cos(x)$

$P_3(0) = a_0 = f(0) = 0 \Rightarrow a_0 = 0$ $\qquad P_3'(0) = a_1 = f'(0) = 1 \Rightarrow a_1 = 1$

$P_3''(0) = 2a_2 = f''(0) = 0 \Rightarrow a_2 = 0$

$P_3'''(0) = 6a_3 = f'''(0) = -1 \Rightarrow a_3 = -\frac{1}{6}$

$P_3(x) = x + \left(-\frac{1}{6}\right)x^3 = x - \frac{1}{6}x^3$ $\sin(0.2) \approx 0.2 - \frac{1}{6}(0.2)^3 = 0.19866\ldots$

### Definition of a Taylor Polynomial

Let's see if we can come up with a general rule that can be used to quickly find a polynomial approximation for any elementary function this is differentiable to the degree of the polynomial.

Our previous examples were all centered at 0, but let's consider a more general case where the polynomial is entered at $x = c$.

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 \ldots a_n(x-c)^n$$

We need to match up traits of our polynomial with our function $f(x)$. $S$o, we need:

$f(c) = P_n(c)$ $\qquad f'(c) = P_n{}'(c)$ $\qquad f''(c) = P_n{}''(c)$ $\qquad \ldots$ $\qquad f^{(n)}(c) = P_n^{(n)}(c)$

Let's start by finding the derivatives of $P_n(x)$:

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 \ldots + a_n(x-c)^n$$

$$P_n'(x) = 0 + a_1 + 2a_2(x-c)^1 + 3a_3(x-c)^2 + 4a_4(x-c)^3 \ldots + na_n(x-c)^{n-1}$$

$$P_n''(x) = 2a_2 + 3 \cdot 2a_3(x-c) + 4 \cdot 3a_4(x-c)^2 + \cdots + n(n-1)a_n(x-c)^{n-2}$$

$$P_n^{(3)}(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x-c) + \cdots + n(n-1)(n-1)a_n(x-c)^{n-3}$$

$$P_n^{(n)}(x) = n(n-1)(n-2)(n-3) \ldots (2)(1)a_n$$

Now, we can begin to find $a_0, a_1, a_2, \ldots, a_n$ by letting $x = c$ in our equations:

$\boldsymbol{f(c) = P_n(c)}$

$f(c) = a_0 + a_1(c-c) + a_2(c-c)^2 + a_3(c-c)^3 + a_4(c-c)^4 \ldots + a_n(c-c)^n = a_0$

$f(c) = a_0$

$\boldsymbol{f'(c) = P_n{}'(c)}$

$f'(c) = a_1 + 2a_2(c-c)^1 + 3a_3(c-c)^2 + 4a_4(c-c)^3 \ldots + na_n(c-c)^{n-1} = a_1$

$f'(c) = a_1$

$\boldsymbol{f''(c) = P_n{}''(c)}$

$f''(c) = 2a_2 + 3 \cdot 2a_3(c-c) + 4 \cdot 3a_4(c-c)^2 + \cdots + n(n-1)a_n(c-c)^{n-2} = 2a_2$

$$f''(c) = 2a_2 \rightarrow a_2 = \frac{f''(c)}{2}$$

$\boldsymbol{f^{(n)}(c) = P_n^{(n)}(c)}$

$$f^{(n)}(c) = n(n-1)(n-2)(n-3)\ldots(2)(1)a_n \rightarrow a_n = \frac{f^{(n)}(c)}{n!}$$

| **Definitions of $n$th Taylor Polynomial and $n$th Macclaurin Polynomial** |
|---|
| If $f$ has n derivatives at c, then the polynomial<br><br>$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$<br>is called the $n$th **Taylor polynomial** for $f$ at $c$.<br><br>If $c = 0$:<br>$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!} + \cdots + \frac{f^{(n)}(0)(x)^n}{n!}$<br><br>is called the $n$th **Maclaurin polynomial** for $f$. |

**Example 4**: Find a 2nd degree Taylor polynomial for $f(x) = \cos(x)$ centered at $x = \frac{\pi}{3}$

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

$$f\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad f'\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad f''\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$P_2(x) = \frac{1}{2} + -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{\left(-\frac{1}{2}\right)\left(x - \frac{\pi}{3}\right)^2}{2!} = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2$$

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ | $f'''(x)$ |
|---|---|---|---|---|
| 1 | 2 | −3 | 1 | −6 |

**Example 5**: The functions $f$ and $g$ are differentiable for all orders $n$. Find a third degree Taylor polynomial for $g(x)$ centered at $x = 1$ where $g(x) = \int_1^x f(t)dt$

$$g(1) = 0 \qquad g'(1) = f(1) = 2 \qquad g''(1) = f'(1) = -3 \qquad g'''(1) = f''(1) = 1$$

$$P_3(x) = 0 + 2(x-1) + \frac{(-3)(x-1)^2}{2!} + \frac{(1)(x-1)^3}{3!} = 2(x-1) - \frac{3}{2}(x-1)^2 + \frac{1}{6}(x-1)^3$$

# Check for Understanding

**Practice 1**: Find a second degree Maclaurin polynomial for $f(x) = e^{2x}$.

$$f(0) = e^{2(0)} = 1 \qquad f'(0) = 2e^{2(0)} = 2 \qquad f''(0) = 4e^{2(0)} = 4$$

$$P_2(x) = 1 + 2x + \frac{4x^2}{2!} = 1 + 2x + 2x^2$$

**Practice 2**: Find a third degree Taylor polynomial for $g(x) = ln(x)$ centered at $x = 1$.

$$g(1) = \ln(1) = 0 \qquad g'(1) = \frac{1}{1} = 1 \qquad g''(1) = -\frac{1}{1^2} = -1 \qquad g'''(1) = \frac{2}{1^3} = 2$$

$$P_3(x) = 0 + (1)(x-1) + \frac{(-1)(x-1)^2}{2!} + \frac{(2)(x-1)^3}{3!} = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

**Practice 3**: A function $h$ has derivatives of all orders at $x = 0$. It is known that $h(0) = 2, h'(0) = -1$, and $h^{(n)}(0) = \dfrac{n^2}{n!}$ for $n \geq 2$. Find a third degree Maclaurin Polynomial for $h(x)$.

$h(0)=2 \qquad h'(0)=-1 \qquad h''(0)=\dfrac{2^2}{2!} \qquad h'''(0)=\dfrac{3^2}{3!}$

$$P_3(x)=2+(-1)(x)+\frac{\left(\frac{2^2}{2!}\right)(x)^2}{2!}+\frac{\left(\frac{3^2}{3!}\right)(x)^3}{3!}=2-x+x^2+\frac{1}{4}x^3$$

**Practice 4**: A function $g$ has derivatives of all orders at $x = 0$. $P_1(x)$ is the first degree Maclaurin polynomial for $g$ about $x = 0$. If $g(0) = 2$ and $P_1(1) = -4$, find $g'(0)$.

$P_1(x)=g(0)+g'(0)x$

$P_1(x)=2+g'(0)x \Rightarrow P_1(1)=2+g'(0)(1)=-4 \Rightarrow g'(0)=-6$

## Debrief and Summary

<table>
<tr><th>ENDURING UNDERSTANDINGS</th><th>KEY TAKEAWAYS</th></tr>
<tr><td>Power series allow us to represent associated functions on an appropriate interval.</td><td><ul><li>A Taylor polynomial allows us to approximate more complicated functions</li><li>In many cases, as the degree of the Taylor polynomial increases, the polynomial will approach the original function over some interval</li><li>Taylor polynomials can be extended to Taylor series and leads to the idea of Power series.</li></ul></td></tr>
<tr><th colspan="2">COMMON ERRORS, MISCONCEPTIONS & PITFALLS</th></tr>
<tr><td colspan="2"><ul><li>To find a Taylor polynomial, we must consider the “center”</li><li>We must find the values of f and its derivatives at the center</li><li>Taylor polynomials are approximations, but we have not learned how “good” they are at approximating our desired function…yet!</li></ul></td></tr>
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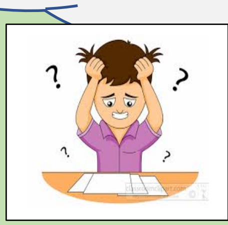
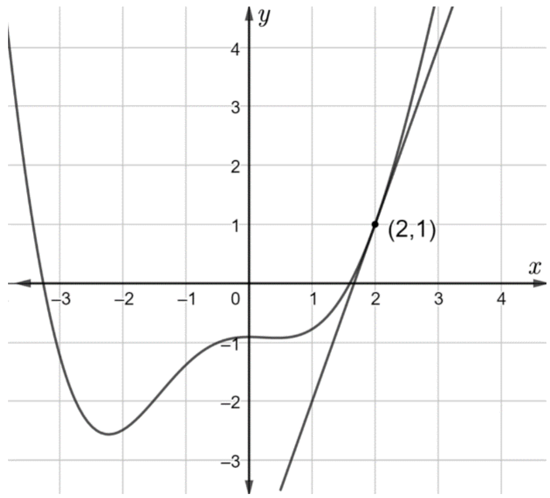


# AP Exam Practice

### AP Practice Problem




**Graph of $f$**

A function $f$ has derivatives of all orders for all values of $x$. A portion of the graph of $f$ is shown above with the line tangent to the graph of $f$ at $x = 2$. Let $g$ be the function defined by $g(x) = 3 + \int_2^x f(t)dt$

(a) Find the second degree Taylor polynomial, $P_2(x)$, for $g(x)$ centered at $x = 2$.

$$g(2) = 3 + \int_2^2 f(t)dt = 3 \qquad g'(2) = f(2) = 1 \qquad g''(2) = f'(2) = 3$$

$$P_2(x) = g(2) + g'(2)(x-2) + \frac{g''(2)(x-2)^2}{2!} = 3 + (x-2) + \frac{3(x-2)^2}{2!} = 3 + (x-2) + \frac{3}{2}(x-2)^2$$

(b) Does $g(x)$ have a local minimum, local maximum, or neither at $x = 2$?

Give a reason for your answer.

$g'(2) = 1 \Rightarrow$ neither because $g'(2) \neq 0$, so $x = 2$ is not a critical point

(c) Consider the geometric series $\sum_{n=1}^{\infty} a_n$ where $a_1 = g'(2)$ and $a_2 = P_2'(x) - 1$.

Find $\sum_{n=1}^{\infty} a_n$ when $x = \frac{13}{6}$.

$$a_1 = g'(2) = 1 \qquad P_2(x) = 3 + (x-2) + \frac{3(x-2)^2}{2!} \qquad a_2 = P_2'(x) - 1 = 3(x-2)$$

$$r = \frac{a_2}{a_1} = \frac{3(x-2)}{1} = 3(x-2) \qquad x = \frac{13}{6} \Rightarrow r = 3\left(\frac{13}{6} - 2\right) = 3\left(\frac{1}{6}\right) = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} a_n = \frac{1}{1 - \frac{1}{2}} = \frac{2}{2-1} = 2$$