AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: 10.5 & 10.9	**Harmonic Series and *p*-Series** **Determining Absolute or Conditional Convergence**	**Date:** April 1, 2020

Warm-Up

Consider the alternating series defined below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

A) Use the alternating series test to show that this series converges when $x = 3$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3)^{2n}}{(2n)!} = 1 - \frac{(3)^2}{2} + \frac{(3)^4}{4!} - \frac{(3)^6}{6!} + \frac{(3)^8}{8!} - \cdots = 1 - \frac{9}{2} + \frac{81}{24} - \frac{729}{720} + \frac{6561}{40320} - \cdots$$

$$a_{n+1} \le a_n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{2n}}{(2n)!} = 0 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (3)^{2n}}{(2n)!} \text{ converges}$$

$$\frac{3^{2n+2}}{(2n+2)!} \le \frac{3^{2n}}{(2n)!} \Rightarrow \frac{3^{2n+2}}{3^{2n}} \le \frac{(2n+2)!}{(2n)!} \Rightarrow 9 \le (2n+2)(2n+1) \Rightarrow 0 \le 4n^2 + 6n - 7 \Rightarrow n \ge 1$$

B) Show that this series converges for all x values where x is a real number.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^{2n}}{(2n)!} = 0$$

$a_{n+1} \le a_n$ when $n > N$ where N is an integer.

$$\frac{x^{2(n+1)}}{(2(n+1))!} \le \frac{x^{2n}}{(2n)!} \Rightarrow \frac{(2n)!}{(2n+2)!} \le \frac{x^{2n}}{x^{2n+2}} \Rightarrow \frac{x^{2n+2}}{x^{2n}} \le \frac{(2n+2)!}{(2n)!} \Rightarrow x^2 \le (2n+2)(2n+1)$$

$0 \le (2n+2)(2n+1) - x^2 \Rightarrow 4n^2 + 6n + (2 - x^2) \ge 0$ $\quad 4n^2 + 6n + (2 - x^2)$ is an open up parabola so when $n >$ the positive zero to the right then $4n^2 + 6n + (2 - x^2) \ge 0$

$$N = \frac{-6 + \sqrt{6^2 - 4(4)(2 - x^2)}}{2(4)} = \frac{-6 + \sqrt{36 - 32 + 16x^2}}{2(4)} = \frac{-6 + \sqrt{4 + 16x^2}}{2(4)} = \frac{-3 + \sqrt{4x^2 + 1}}{4}$$

N can be found for any value of x.

C) Consider the function $f(x)$ where $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Determine if $f(x)$ has a relative minimum, relative maximum or neither at $x = 0$.

Give a reason for your answer. $f'(0) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{(0)^2}{2} + \frac{(0)^4}{4!} - \frac{(0)^6}{6!} + \cdots = 1 \Rightarrow$ Neither

Lesson Overview

WHAT WE ARE GOING TO DO

- **Introduce a very important and quite common series, the *p*-series.**
- **Determine the conditions in which a *p*-series will converge or diverge.**
- **Explain the difference between conditional convergence and absolute convergence.**

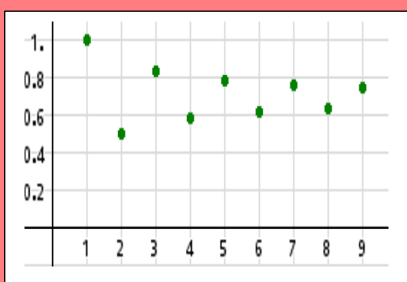
WHAT YOU SHOULD ALREADY KNOW

- **Solid understanding of the concepts *"converge"* and *"diverge"***
- **Techniques for evaluating a limit**
- **Alternating Series Test for Convergence**

WHAT YOU WILL BE ABLE TO DO

Given the following series: $\sum_{n=1}^{\infty} \frac{\sin\left[\left(2n-1\right)\cdot \pi/2\right]}{n}$.

<u>Question</u>: Does this series, conditionally converge, absolutely converge or diverge?



Guided Practice

Let's refresh back to the Alternating Series Test for a moment. Each of the following series below can be easily shown to converge by meeting the conditions of the Alternating Series Test.

$$\sum^{\infty} \frac{(-1)^{n+1}}{n} \qquad \sum^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \qquad \sum^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Topic 10.5 in AP Calculus BC introduces a new series that is quite common, the *p*-series whose convergence/divergence is determined using the information in the box below.

CONVERGENCE OF A *p*-SERIES

The *p*-series is defined by the following where p is a positive real number.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

1. converges if $p > 1$, and

2. diverges if $0 < p \le 1$.

Example 1: Determine if the following series converge or diverge using the *p*-series test. Identify any value(s) for *p*.

a.) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \Rightarrow p = \frac{1}{3}$ diverges because $0 < p \le 1$	**b.)** $\sum_{n=1}^{\infty} \frac{-2}{n^3}$ $\sum_{n=1}^{\infty} \frac{-2}{n^3} = -2\sum_{n=1}^{\infty} \frac{1}{n^3} \Rightarrow p = 3$ converges because $p > 1$	**c.)** $\sum_{n=1}^{\infty} n^{-2} \cdot \sqrt{n}$ $\sum_{n=1}^{\infty} n^{-2} \cdot \sqrt{n} = \sum_{n=1}^{\infty} n^{-2} \cdot n^{1/2} = \sum_{n=1}^{\infty} n^{-3/2}$ $= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \Rightarrow p = \frac{3}{2}$ converges because $p > 1$

d.) $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{n^e} \right)$ $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{n^e} \right) = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{1}{n^e}$ $p = 3$ and $p = e$ Both series converge because $p > 1$	**e.)** $\sum_{n=1}^{\infty} \sqrt[3]{n^4}$ $\sum_{n=1}^{\infty} \sqrt[3]{n^4} = \sum_{n=1}^{\infty} n^{4/3} \Rightarrow$ not a $p-$series so convergence or divergence can not be determined by $p-$series test

DEFINITION of ABSOLUTE and CONDITIONAL CONVERGENCE

1. $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

2. $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

From the previous lesson on the Alternating Series Test, we noticed that if a series is alternating, then it is "easier" for the series to converge. When working with an alternating series, or any series that has both positive and negative terms, it is natural to wonder if the series converged BECAUSE it was alternating or if it would have converged regardless of the alternating component.

Consider the alternating series from the beginning of this lesson: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

We have already determined that this series converges by the alternating series test. Would this series still converge if it was not alternating?

Well, we certainly know the answer to that question as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the *p*-series test where $p = 1$.

What this is all saying is that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ **is conditionally convergent** because

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges but } \sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

These two series above are important and have special names.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series ($p = 1$)

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is the alternating harmonic series

THE HARMONIC SERIES

The harmonic series is simply a special case of a *p*-series where $p = 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Did you know the harmonic series has a close relationship between string instruments and the notes that can be played on them?

Example 2: The Kitchen Sink of Alternating Series
Determine if the following series are absolutely, conditionally convergent or divergent.

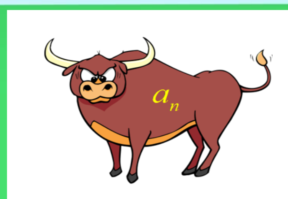
a.) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$ $\sum_{n=1}^{\infty}\left\|\frac{(-1)^n}{n^{3/2}}\right\|=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\Rightarrow$ converges, $p-$series with $p=\frac{3}{2}>1.$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$ converges absolutely	**b.)** $\sum_{n=1}^{\infty}\frac{(-1)^n}{8}$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{8}\Rightarrow$ diverges $\lim_{n\to\infty}\frac{1}{8}=\frac{1}{8}\neq 0$ $\sum_{n=1}^{\infty}\left\|\frac{(-1)^n}{8}\right\|=\sum_{n=1}^{\infty}\frac{1}{8}\Rightarrow S_n=\frac{1}{8}n\Rightarrow$ diverges $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(\frac{1}{8}n\right)\to\infty$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{8}$ is divergent
c.) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0 \qquad \sqrt{n+1}\geq\sqrt{n}\Rightarrow\frac{1}{\sqrt{n+1}}\leq\frac{1}{\sqrt{n}}$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by alternating series test. $\sum_{n=1}^{\infty}\left\|\frac{(-1)^n}{\sqrt{n}}\right\|=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\ \Rightarrow$ divergent $p=\frac{1}{2}\leq 1$ Conditionally convergent	**d.)** $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot\sqrt[8]{n}}{2n}$ $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot\sqrt[8]{n}}{2n}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^{1/8}}{n}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{7/8}}$ $\lim_{n\to\infty}\frac{1}{n^{7/8}}=0 \qquad (n+1)^{7/8}\geq n^{7/8}\Rightarrow\frac{1}{(n+1)^{7/8}}\leq\frac{1}{n^{7/8}}$ converges by alternating series test. $\sum_{n=1}^{\infty}\left\|\frac{(-1)^{n-1}}{n^{7/8}}\right\|=\sum_{n=1}^{\infty}\frac{1}{n^{7/8}}\Rightarrow$ divergent $p-$series $p=\frac{7}{8}\leq 1$ Conditionally convergent

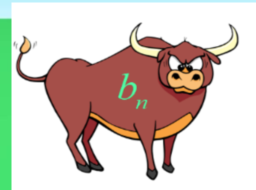
e.) $\sum_{n=1}^{\infty}(-1)^n \frac{2n-3}{5n+2}$ $\sum_{n=1}^{\infty}(-1)^n \frac{2n-3}{5n+2}$ $\sum_{n=1}^{\infty}\left\lvert(-1)^n \frac{2n-3}{5n+2}\right\rvert$ $\Rightarrow \lim_{n\to\infty}\frac{2n-3}{5n+2}=\frac{2}{5}\neq 0$ Both divergent	**f.)** $\sum_{n=1}^{\infty}4\left(-\frac{1}{3}\right)^n$ $\sum_{n=1}^{\infty}4\left(-\frac{1}{3}\right)^n = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n}$ $\sum_{n=1}^{\infty}\left\lvert\frac{(-1)^n}{3^n}\right\rvert=\sum_{n=1}^{\infty}\frac{1}{3^n}$ $S_n=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots+\frac{1}{3^n}$ $\frac{1}{3}S_n=\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots+\frac{1}{3^n}+\frac{1}{3^{n+1}}$ $S_n-\frac{1}{3}S_n=\frac{1}{3}-\frac{1}{3^{n+1}}\Rightarrow\frac{2}{3}S_n=\frac{1}{3}-\frac{1}{3^{n+1}}$ $S_n=\frac{3}{2}\left(\frac{1}{3}-\frac{1}{3^{n+1}}\right)$ $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[\frac{3}{2}\left(\frac{1}{3}-\frac{1}{3^{n+1}}\right)\right]=\frac{3}{2}\left(\frac{1}{3}-0\right)=\frac{1}{2}$ Absolutely convergent
g.) $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{\cos(n)}$ $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{\cos(n)}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\cos(n)}$ $\sum_{n=1}^{\infty}\left\lvert\frac{(-1)^n}{\cos(n)}\right\rvert=\sum_{n=1}^{\infty}\frac{1}{\cos(n)}$ $\Rightarrow\lim_{n\to\infty}\frac{1}{\cos(n)}\neq 0$ divergent by nth term test	**h.)** $\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{n^2+5n+1}$ $\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{n^2+5n+1}$ $\sum_{n=1}^{\infty}\left\lvert(-1)^n\frac{e^n}{n^2+5n+1}\right\rvert$ $\Rightarrow\lim_{n\to\infty}\frac{e^n}{n^2+5n+1}\to\infty$ both are divergent by nth term test

Check for Understanding

Practice 1: Alternating Series Roundup

For each series below, k is a constant. Use the information about the given series to answer the following questions.





$a_n = \dfrac{1}{n^k}$	$b_n = (a_n)^2$	$c_n = \dfrac{1}{n^{4k-2}}$	$d_n = \dfrac{1}{n^{3-2k}}$	$e_n = \dfrac{1}{n^{5k-1}}$	$f_n = \dfrac{1}{\sqrt{n^{\left(k+\frac{8}{5}\right)}}}$

a.) Find a value of k such that $\sum_{n=1}^{\infty}(-1)^n a_n$ is a conditionally convergent series and $\sum_{n=1}^{\infty}(-1)^n b_n$ is absolutely convergent.

$$\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^k} \qquad \sum_{n=1}^{\infty}(-1)^n b_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k}}$$

Both are convergent by alternating series test when $k > 0$

$$\sum_{n=1}^{\infty}\left|(-1)^n a_n\right| = \sum_{n=1}^{\infty}\frac{1}{n^k} \quad \text{conditionally convergent } \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^k} \text{ divergent } p-\text{series} \Rightarrow p = k \le 1$$

$$\sum_{n=1}^{\infty}\left|(-1)^n b_n\right| = \sum_{n=1}^{\infty}\frac{1}{n^{2k}} \quad \text{absolutely convergent } \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^{2k}} \text{ convergent } p-\text{series} \Rightarrow p = 2k > 1 \Rightarrow k > \frac{1}{2}$$

$$\frac{1}{2} < k \le 1$$

b.) Find the maximum value of k such that $\sum_{n=1}^{\infty}(-1)^n c_n$ is a conditionally convergent series and $\sum_{n=1}^{\infty}(-1)^n d_n$ is absolutely convergent.

$$\sum_{n=1}^{\infty}(-1)^n c_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{4k-2}} \qquad \sum_{n=1}^{\infty}(-1)^n d_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3-2k}}$$

Both are convergent byalternating series test when $4k-2>0 \Rightarrow k > \frac{1}{2}$ and $3-2k>0 \Rightarrow k < \frac{3}{2}$

$$\sum_{n=1}^{\infty}\left|(-1)^n c_n\right| = \sum_{n=1}^{\infty}\frac{1}{n^{4k-2}} \quad \text{conditionally convergent } \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^{4k-2}} \text{ divergent } p-\text{series} \Rightarrow 4k-2 \le 1 \Rightarrow k \le \frac{3}{4}$$

$$\sum_{n=1}^{\infty}\left|(-1)^n d_n\right| = \sum_{n=1}^{\infty}\frac{1}{n^{3-2k}} \quad \text{absolutely convergent } \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^{3-2k}} \text{ convergent } p-\text{series} \Rightarrow 3-2k > 1 \Rightarrow k < 1$$

$$\frac{1}{2} < k \le \frac{3}{4} \Rightarrow k = \frac{3}{4} \text{ is the maximum value}$$

c.) If $\sum_{n=1}^{\infty}(-1)^n e_n$ is absolutely convergent, determine if $\sum_{n=1}^{\infty}(-1)^n f_n$ is absolutely convergent, conditionally convergent or divergent.

$\sum_{n=1}^{\infty}\left|(-1)^n e_n\right| = \sum_{n=1}^{\infty}\frac{1}{n^{5k-1}}$ absolutely convergent $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^{5k-1}}$ convergent $p-$series $\Rightarrow 5k-1>1 \Rightarrow k>\frac{2}{5}$

$\sum_{n=1}^{\infty}\left|(-1)^n f_n\right| = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}k+\frac{4}{5}}}$ $k>\frac{2}{5} \Rightarrow \frac{1}{2}k+\frac{4}{5}>1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}k+\frac{4}{5}}}$ convergent $p-$series with $p>1$

$\sum_{n=1}^{\infty}(-1)^n f_n$ is absolutely convergent.

d.) If $\sum_{n=1}^{\infty}(-1)^n c_n$ diverges, which alternating series must be absolutely convergent?

$\sum_{n=1}^{\infty}(-1)^n c_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{4k-2}}$ divergent $\Rightarrow \lim_{n\to\infty}\frac{1}{n^{4k-2}} \neq 0 \Rightarrow \lim_{n\to\infty} n^{4k-2} = 0 \Rightarrow 4k-2<0 \Rightarrow k<\frac{1}{2}$

$k<\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^k}$ converges by AST $\sum_{n=1}^{\infty}\frac{1}{n^k}$ divergent $p-$series $\Rightarrow k<\frac{1}{2}$

$k<\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}(-1)^n b_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k}}$ converges by AST $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$ divergent $p-$series $\Rightarrow k<\frac{1}{2} \Rightarrow 2k<1$

$k<\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}(-1)^n d_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3-2k}}$ converges by AST $\sum_{n=1}^{\infty}\frac{1}{n^{3-2k}}$ convergent $p-$series $\Rightarrow k<\frac{1}{2} \Rightarrow 3-2k>2$

$k<\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}(-1)^n e_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{5k-1}}$ converges by AST $\sum_{n=1}^{\infty}\frac{1}{n^{5k-1}}$ divergent $p-$series $\Rightarrow k<\frac{1}{2} \Rightarrow 5k-1<\frac{3}{2}$

$k<\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}(-1)^n f_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\frac{1}{2}k+\frac{4}{5}}}$ converges by AST $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}k+\frac{4}{5}}}$ divergent $p-$series $\Rightarrow k<\frac{1}{2} \Rightarrow \frac{1}{2}k+\frac{4}{5}<1$

$\sum_{n=1}^{\infty}(-1)^n d_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3-2k}}$ is absolutely convergent when $k<\frac{1}{2}$.

Practice 2: More Alternating Series

Determine if the following series are absolutely, conditionally convergent or divergent.

a.) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^2}+\frac{1}{n^5}\right)$

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^2}+\frac{1}{n^5}\right)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}$$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow \text{absolutely convergent}$$

$p-\text{series } p=2>1$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^5}\right|=\sum_{n=1}^{\infty}\frac{1}{n^5}\Rightarrow \text{absolutely convergent}$$

$p-\text{series } p=5>1$

$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^2}+\frac{1}{n^5}\right)$ is absolutely convergent because it is the sum of two absolutely convergent series

b.) $\sum_{n=1}^{\infty}\frac{(-\pi)^n}{e^{n+1}}$

$$\sum_{n=1}^{\infty}\frac{(-\pi)^n}{e^{n+1}}=\sum_{n=1}^{\infty}\frac{(-1)^n\pi^n}{e^{n+1}}$$

$$\lim_{n\to\infty}\frac{\pi^n}{e^{n+1}}=\frac{1}{e}\lim_{n\to\infty}\frac{\pi^n}{e^n}=\frac{1}{e}\lim_{n\to\infty}\left(\frac{\pi}{e}\right)^n\to\infty$$

because $\frac{\pi}{e}>1$ divergent by nth term test

c.) $\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt[3]{n}}{\sqrt{n}}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt[3]{n}}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/6}}$$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^{1/6}}\right|=\sum_{n=1}^{\infty}\frac{1}{n^{1/6}}\Rightarrow \text{ divergent } p-\text{series } \; p=\frac{1}{6}\le 1$$

$$\lim_{n\to\infty}\frac{1}{n^{1/6}}=0 \qquad \frac{1}{(n+1)^{1/6}}\le n^{1/6}$$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/6}}$ is convergent by alternating series test.

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/6}}$ is conditionally convergent

d.) $\sum_{n=1}^{\infty}a_n$ where $a_n=\begin{cases}\frac{1}{n+3}, & n \text{ is odd}\\ -\frac{1}{n+3}, & n \text{ is even}\end{cases}$

$$\sum_{n=1}^{\infty}a_n \text{ where } a_n=\begin{cases}\frac{1}{n+3}, & n \text{ is odd}\\ -\frac{1}{n+3}, & n \text{ is even}\end{cases}$$

$$\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{1}{n+3}=\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots=\sum_{n=4}^{\infty}\frac{1}{n}$$

$\sum_{n=4}^{\infty}\frac{1}{n}$ is a divergent $p-$series $\; p=1\le 1$

$$\sum_{n=1}^{\infty}a_n=\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}\cdots=\sum_{n=4}^{\infty}\frac{(-1)^{n-1}}{n}$$

$\sum_{n=4}^{\infty}\frac{(-1)^{n-1}}{n}$ is a convergent alternating harmonic series

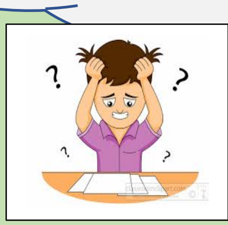
$\sum_{n=1}^{\infty}a_n$ is conditionally convergent.

Debrief and Summary

ENDURING UNDERSTANDINGS	KEY TAKEAWAYS
Applying limits may allow us to determine the finite sum of infinitely many terms.	- If $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$ - $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \lvert a_n \rvert$ converges. - $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} \lvert a_n \rvert$ diverges.

COMMON ERRORS, MISCONCEPTIONS & PITFALLS

- Be sure to simplify exponents when working with *p* - series.
- Alternating series can converge in two different ways. It is best to start by determining the behavior of the series of the absolute value of the nth-term expression and then applying the AST.



AP Exam Practice

Let $a(n)=\dfrac{1}{n^{k+1}}$ where k is a constant

(a) For $k=\dfrac{1}{2}$, use the alternating series test to show that $\sum_{n=1}^{\infty}(-1)^n a(n)$ converges. Determine if this series converges conditionally or converges absolutely. Explain your reasoning.

$$k=\frac{1}{2}\Rightarrow\sum_{n=1}^{\infty}(-1)^n a(n)=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty}\frac{1}{n^{3/2}}\to\frac{1}{\infty}\Rightarrow\lim_{n\to\infty}\frac{1}{n^{3/2}}=0 \qquad \frac{1}{(n+1)^{3/2}}\le\frac{1}{n^{3/2}}\Rightarrow a(n+1)\le a(n)$$

$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{3/2}}$ is a convergent alternating series

$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n^{3/2}}\right|=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ convergent $p-$series with $p=\frac{3}{2}>1$

$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{3/2}}$ converges absolutely

(b) Let $b(n)=a(\sqrt{n})$. Find all integer values of k such that $\sum_{n=1}^{\infty}(-1)^n b(n)$ converges conditionally.

$$\sum_{n=1}^{\infty}(-1)^n b(n)=\sum_{n=1}^{\infty}(-1)^n\frac{1}{\left(\sqrt{n}\right)^{k+1}}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{\frac{1}{2}(k+1)}}$$

converges conditionally $\Rightarrow\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{\frac{1}{2}(k+1)}}$ converges but $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}(k+1)}}$ does not converge.

$$\lim_{n\to\infty}\frac{1}{n^{\frac{1}{2}(k+1)}}=0\Rightarrow\lim_{n\to\infty}n^{\frac{1}{2}(k+1)}\to\infty\Rightarrow\frac{1}{2}(k+1)>0\Rightarrow k>-1 \qquad \frac{1}{(n+1)^{\frac{1}{2}(k+1)}}\le\frac{1}{n^{\frac{1}{2}(k+1)}}\Rightarrow b(n+1)\le b(n)$$

$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}(k+1)}}$ is a divergent $p-$series with $p=\frac{1}{2}(k+1)\le 1\Rightarrow k\le 1$

$-1<k\le 1\Rightarrow k=0,1$

(c) Let $c(n)=a(n^{-2k})$. Show that there is no real value of k such that $\sum_{n=1}^{\infty}c(n)$ is the harmonic series.

$$\sum_{n=1}^{\infty}c(n)=\sum_{n=1}^{\infty}\frac{1}{\left(n^{-2k}\right)^{k+1}}=\sum_{n=1}^{\infty}\frac{1}{n^{-2k(k+1)}}$$

Harmonic series $\Rightarrow -2k(k+1)=1$

$0=2k^2+2k+1\Rightarrow k=\dfrac{-2\pm\sqrt{2^2-4(2)(1)}}{2(2)}=\dfrac{-2\pm\sqrt{-4}}{4}$ which is not a real number