AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: 10.5 & 10.9	**Harmonic Series and *p*-Series** **Determining Absolute or Conditional Convergence**	**Date:** April 1, 2020

Warm-Up

Consider the alternating series defined below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

A) Use the alternating series test to show that this series converges when $x = 3$.

B) Show that this series converges for all x values where x is a real number.

C) Consider the function $f(x)$ where $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

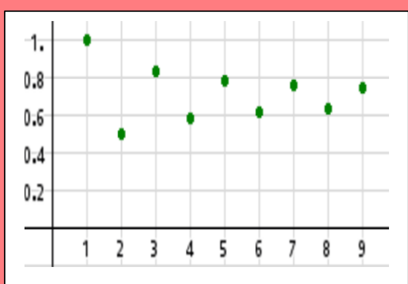
Determine if $f(x)$ has a relative minimum, relative maximum or neither at $x = 0$.
Give a reason for your answer.

Lesson Overview

WHAT WE ARE GOING TO DO	WHAT YOU SHOULD ALREADY KNOW
• Introduce a very important and quite common series, the *p*-series. • Determine the conditions in which a *p*-series will converge or diverge. • Explain the difference between conditional convergence and absolute convergence.	• Solid understanding of the concepts "*converge*" and "*diverge*" • Techniques for evaluating a limit • Alternating Series Test for Convergence

WHAT YOU WILL BE ABLE TO DO

Given the following series: $\sum_{n=1}^{\infty} \frac{\sin\left[(2n-1)\cdot \pi/2\right]}{n}$.



Question: Does this series, conditionally converge, absolutely converge or diverge?

Guided Practice

Let's refresh back to the Alternating Series Test for a moment. Each of the following series below can be easily shown to converge by meeting the conditions of the Alternating Series Test.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Topic 10.5 in AP Calculus BC introduces a new series that is quite common, the *p*-series whose convergence/divergence is determined using the information in the box below.

CONVERGENCE OF A *p*-SERIES

The *p*-series is defined by the following where p is a positive real number.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

1. converges if $p > 1$, and

2. diverges if $0 < p \leq 1$.

Example 1: Determine if the following series converge or diverge. Identify any value(s) for p.

a.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$	**b.)** $\displaystyle\sum_{n=1}^{\infty} \frac{-2}{n^3}$	**c.)** $\displaystyle\sum_{n=1}^{\infty} n^{-2} \cdot \sqrt{n}$

d.) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{n^e} \right)$	**e.)** $\displaystyle\sum_{n=1}^{\infty} \sqrt[3]{n^4}$

DEFINITION of ABSOLUTE and CONDITIONAL CONVERGENCE

1. $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

2. $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

From the previous lesson on the Alternating Series Test, we noticed that if a series is alternating, then it is "easier" for the series to converge. When working with an alternating series, or any series that has both positive and negative terms, it is natural to wonder if the series converged BECAUSE it was alternating or if it would have converged regardless of the alternating component.

Consider the alternating series from the beginning of this lesson: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

We have already determined that this series converges by the alternating series test. Would this series still converge if it was not alternating?

Well, we certainly know the answer to that question as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the *p*-series test where $p = 1$.

What this is all saying is that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ **is conditionally convergent** because

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

These two series above are important and have special names.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series ($p = 1$)

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is the alternating harmonic series

THE HARMONIC SERIES

The harmonic series is simply a special case of a *p*-series where $p = 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Did you know the harmonic series shares a close relationship between stringed instruments and the notes that can be played on them?

Example 2: The Kitchen Sink of Alternating Series

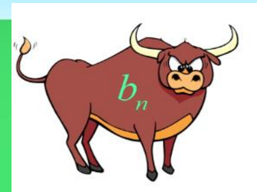
Determine if the following series are absolutely, conditionally convergent or divergent.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$	**b.)** $\sum_{n=1}^{\infty} \frac{(-1)^n}{8}$
c.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$	**d.)** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot \sqrt[8]{n}}{2n}$
e.) $\sum_{n=1}^{\infty} (-1)^n \frac{2n-3}{5n+2}$	**f.)** $\sum_{n=1}^{\infty} 4\left(-\frac{1}{3}\right)^n$
g.) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\cos(n)}$	**h.)** $\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{n^2+5n+1}$

Check for Understanding

Practice 1: Alternating Series Roundup

For each series below, k is a constant. Use the information about the given series to answer the following questions.



$a_n = \dfrac{1}{n^k}$	$b_n = (a_n)^2$	$c_n = \dfrac{1}{n^{4k-2}}$	$d_n = \dfrac{1}{n^{3-2k}}$	$e_n = \dfrac{1}{n^{5k-1}}$	$f_n = \dfrac{1}{\sqrt{n^{\left(k+\frac{8}{5}\right)}}}$

a.) Find a value of k such that $\sum_{n=1}^{\infty}(-1)^n a_n$ is a conditionally convergent series and $\sum_{n=1}^{\infty}(-1)^n b_n$ is absolutely convergent.

b.) Find the maximum value of k such that $\sum_{n=1}^{\infty}(-1)^n c_n$ is a conditionally convergent series and $\sum_{n=1}^{\infty}(-1)^n d_n$ is absolutely convergent.

c.) If $\sum_{n=1}^{\infty}(-1)^n e_n$ is absolutely convergent, determine if $\sum_{n=1}^{\infty}(-1)^n f_n$ is absolutely convergent, conditionally convergent or divergent.

d.) If $\sum_{n=1}^{\infty}(-1)^n c_n$ diverges, which alternating series must be absolutely convergent?

Practice 2: More Alternating Series

Determine if the following series are absolutely, conditionally convergent or divergent.

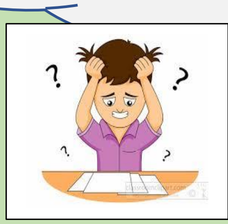
a.) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^2}+\frac{1}{n^5}\right)$	**b.)** $\displaystyle\sum_{n=1}^{\infty}\frac{(-\pi)^n}{e^{n+1}}$
c.) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt[3]{n}}{\sqrt{n}}$	**d.)** $\displaystyle\sum_{n=1}^{\infty}a_n$ where $a_n=\begin{cases}\dfrac{1}{n+3}, & n \text{ is odd}\\ -\dfrac{1}{n+3}, & n \text{ is even}\end{cases}$

Debrief and Summary

ENDURING UNDERSTANDINGS	KEY TAKEAWAYS
Applying limits may allow us to determine the finite sum of infinitely many terms.	- If $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$ - $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \lvert a_n \rvert$ converges. - $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} \lvert a_n \rvert$ diverges.

COMMON ERRORS, MISCONCEPTIONS & PITFALLS

- Be sure to simplify exponents when working with *p* - series.
- Alternating series can converge in two different ways. It is best to start by determining the behavior of the series of the absolute value of the nth-term expression and then applying the AST.



AP Exam Practice

AP Practice Problem

$$\text{Let } a(n) = \frac{1}{n^{k+1}} \text{ where } k \text{ is a constant}$$

(**a**) For $k = \frac{1}{2}$, use the alternating series test to show that $\sum_{n=1}^{\infty}(-1)^n a(n)$ converges. Determine if this series converges conditionally or converges absolutely. Explain your reasoning.

(**b**) Let $b(n) = a(\sqrt{n})$. Find all integer values of k such that $\sum_{n=1}^{\infty}(-1)^n b(n)$ converges conditionally.

(**c**) Let $c(n) = a(n^{-2k})$. Show that there is no real value of k such that $\sum_{n=1}^{\infty} c(n)$ is the harmonic series.