AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: 10.7	**Alternating Series Test for Convergence**	**Date:** March 31, 2020

Warm-Up

Determine, if possible, if each of the given series diverges based on the nth term test for divergence

a) $\displaystyle\sum_{n=0}^{\infty} 5^n$

$$\lim_{n\to\infty}\left(5^n\right) \to \infty \Rightarrow \sum_{n=0}^{\infty} 5^n \text{ diverges}$$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{2n!}{1-3n!}$

$$\lim_{n\to\infty}\frac{2n!}{1-3n!} = -\frac{2}{3} \Rightarrow \sum_{n=0}^{\infty}\frac{2n!}{1-3n!} \text{ diverges}$$

c) $\displaystyle\sum_{n=0}^{\infty}\left(3^{1-n}\cdot 2^{1+n}\right)$

$$\lim_{n\to\infty}\left(3^{1-n}\cdot 2^{1+n}\right) = \lim_{n\to\infty}\left(3\cdot 3^{-n}\cdot 2\cdot 2^{n}\right) = \lim_{n\to\infty}\left(6\cdot\left(\frac{2}{3}\right)^{n}\right) = 0$$

$$\sum_{n=0}^{\infty}\left(3^{1-n}\cdot 2^{1+n}\right) \text{ may converge or diverge}$$

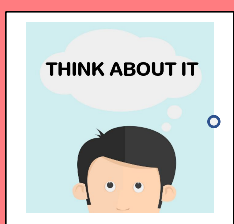
Lesson Overview

WHAT WE ARE GOING TO DO	WHAT YOU SHOULD ALREADY KNOW
• Solidify our understanding of partial sums and convergence vs. divergence of a series • Understand and use the Alternating Series Test when applicable	• The nth term test for divergence • How to evaluate a limit at infinity

WHAT YOU WILL BE ABLE TO DO

Consider the four series given below. Which of these series can be shown to converge using the alternating series test?

$$I.\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad II.\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n}{2n+3} \qquad III.\ \sum_{n=0}^{\infty} \frac{\cos(\pi n)}{n!} \qquad IV.\ \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n+2}$$



Question: Which of these meet the conditions needed to apply the alternating series test?

Guided Practice

Let's briefly review some of the important concepts and vocabulary needed for Unit 10…

Sequence	Series
a_n	$\sum_{n=1}^{\infty} a_n$
Partial Sum: $S_n = a_1 + a_2 + a_3 + \cdots + a_n$	
A series converges if: $\lim_{n\to\infty} S_n = L$	A series diverges if: $\lim_{n\to\infty} S_n \neq L$
n-th term test for divergence If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges	

Given a series, we are interested in knowing if the series converges or diverges. If a series converges, we are often interested in what the series converges to…although this is not always possible (especially in an entry level calculus course).

Example 1: Consider the following series below. Determine if the series converge or diverge.

$a)\ \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots$

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0 \Rightarrow \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \text{ may diverge or converge}$$

$$S_1 = \frac{1}{2} \qquad S_2 = \frac{1}{2}+\frac{1}{4}=\frac{3}{4} \qquad S_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8} \Rightarrow S_n = \frac{2^n-1}{2^n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{2^n-1}{2^n} = 1 \Rightarrow \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \text{ converges}$$

$b)\ \sum_{n=1}^{\infty}(-1)^n$

$$\lim_{n\to\infty}(-1)^n \neq 0 \Rightarrow \sum_{n=0}^{\infty}(-1)^n \Rightarrow \text{ diverges}$$

$c)\ \sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}}$$

$$+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}>\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{2}}+\cdots$$

$$S_2 = \frac{3}{2} \qquad S_4 > 2 \qquad S_8 > \frac{5}{2} \qquad S_{16} > 3$$

$$\Rightarrow S_{2^n} > \frac{3}{2}+\left(\frac{1}{2}\right)(n-1) \Rightarrow \lim_{n\to\infty} S_{2^n} \to \infty$$

$$\sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges}$$

Alternating Series

An alternating series is a series whose terms alternate in signs.

A few examples of alternating series include:

a) $\frac{7}{3}-\frac{1}{2}+2-\frac{1}{4}+\frac{5}{3}-\frac{1}{8}+\frac{4}{3}-\frac{1}{16}+\cdots$

b) $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$

c) $\sum_{n=0}^{\infty}\left(\frac{-2}{3}\right)^n$

d) $\sum_{n=2}^{\infty}\frac{\cos(\pi n)}{n!}$

Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{and} \qquad \sum_{n=1}^{\infty}(-1)^{n-1} a_n$$

converge if the following conditions are both met:

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ for all $n > N$ where N is an integer

Example 2: Determine the convergence or divergence of $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} \to \frac{1}{\infty} \Rightarrow \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0$$

$a_{n+1} = \frac{1}{\sqrt{n+2}} \le a_n = \frac{1}{\sqrt{n+1}}$ $n \ge 0$ because the denominators are getting larger $\Rightarrow$ fraction is smaller

$\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges

Example 3: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - 6n + 10}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2-6n+10} \to \frac{1}{\infty} \Rightarrow \lim_{n\to\infty} \frac{1}{n^2-6n+10} = 0$

$\frac{d}{dn}\left(n^2-6n+10\right) = 2n-6 > 0 \Rightarrow n > 3 \Rightarrow$ denominators are increasing $\Rightarrow$ fraction is smaller

$a_1 = \frac{1}{5},\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{1},\ a_4 = \frac{1}{2}, a_5 = \frac{1}{5},\ a_6 = \frac{1}{10}$

$a_{n+1} \le a_n$ when $n > 3$ $\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^2-6n+10}$ converges.

Example 4: Consider the series given below. For each, determine if the altnernating series test can be applied. If not, explain why not.

a) $\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{n!}$

Yes

$\cos(0) = 1 \qquad \cos(\pi) = -1$

$\cos(2\pi) = 1 \qquad \cos(3\pi) = -1$

$\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Rightarrow$ alternating

$a_n = \frac{1}{n!} > 0 \qquad (n+1)! > n! \Rightarrow \frac{1}{(n+1)!} < \frac{1}{n!}$

$a_{n+1} \le a_n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n!} = 0$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{5n-3}$

NO

$a_n = \frac{1}{5n-3} > 0\ ,\ n \ge 1$

$(-1)^{2(1)-1} = (-1) \qquad (-1)^{2(2)-1} = (-1)^3 = -1$

$(-1)^{2(3)-1} = (-1)^5 = -1 \Rightarrow (-1)^{2n-1} = (-1)^{odd} = -1$

Not alternating

c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot n}{3n+1}$

NO

$(-1)^{0+1} = -1 \qquad (-1)^{1+1} = (-1)^2 = 1$

$(-1)^{2+1} = (-1)^3 = -1 \Rightarrow$ alternating

$\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \ne 0$

d) $\sum_{n=1}^{\infty} a_n$ where $a_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd} \\ -\frac{1}{n^2}, & \text{if } n \text{ is even} \end{cases}$

NO

$\sum_{n=1}^{\infty} a_n = \frac{1}{1} - \frac{1}{4} + \frac{1}{3} - \frac{1}{16} + \frac{1}{5} - \frac{1}{36} \cdots$

Alternating and $a_n > 0$

$\lim_{n\to\infty} a_n = \begin{cases} \lim_{n\to\infty} \frac{1}{n} = 0 \\ \lim_{n\to\infty} \frac{1}{n^2} = 0 \end{cases} = 0$

$a_2 < a_3,\ a_4 < a_5 \Rightarrow a_{n+1} \not\le a_n$ for all $n > N$

Check for Understanding

Practice 1: Show that the following series converges using the alternating series test

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \cdots$$

Alternating and $a_n > 0$ $\quad \lim_{n\to\infty} \frac{1}{(2n)!} \to \frac{1}{\infty} \Rightarrow \lim_{n\to\infty} \frac{1}{(2n)!} = 0$

$$(2(n+1))! > (2n)! \Rightarrow \frac{1}{(2(n+1))!} < \frac{1}{(2n)!}$$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ converges

Practice 2: There are six series listed below. For each of the six series, determine which of the two categories below they fall into.

Alternating Series Test does not apply	Converges by the Altnerating Series Test
C. $\sum_{n=0}^{\infty} \frac{(-1)^{2n}}{4^n}$	A. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$
B. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2n-1}{7n+3}$	D. $\sum_{n=1}^{\infty} \cos(\pi n) \cdot n^{-1}$
	E. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n^2+2}$
	F. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt[3]{n}}{n}$

Practice 3: Each statement below if false. Correct each statement to create a true statement.

For Statements 1 – 3: Let $a_n > 0$

Statement 1: If $a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

If $a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

Statement 2: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

If $a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

Statement 3: If $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges, then $\lim_{n\to\infty} a_n = 0$

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges, if $\lim_{n\to\infty} a_n = 0$

Statement 4: Consider the series $\sum_{n=1}^{\infty} b_n$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\lim_{n\to\infty} b_n \neq 0$

$\sum_{n=1}^{\infty} b_n$ diverges, if $\lim_{n\to\infty} b_n \neq 0$

Debrief and Summary

ENDURING UNDERSTANDING	KEY TAKEAWAY
Applying limits may allow us to determine the finite sum of infinitely many terms.	For $a_n > 0$, the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges if 1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \leq a_n$ for all $n > N$ where N is an integer

COMMON ERRORS, MISCONCEPTIONS & PITFALLS

- The alternating series test is for CONVERGENCE.
- For any series, if $\lim_{n\to\infty} a_n \neq 0$, the series diverges
- A divergent series does **not** necessarily imply $\lim_{n\to\infty} S_n \to \infty$

AP Exam Practice

Consider the altnerating series defined below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

A) Use the alternating series test to show that this series converges when $x = 3$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3)^{2n}}{(2n)!} = 1 - \frac{(3)^2}{2} + \frac{(3)^4}{4!} - \frac{(3)^6}{6!} + \frac{(3)^8}{8!} - \cdots 1 - \frac{9}{2} + \frac{81}{24} - \frac{729}{720} + \frac{6561}{40320} - \cdots$$

$a_{n+1} < a_n$ when $n > 1$ $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{2n}}{(2n)!} = 0 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (3)^{2n}}{(2n)!}$ converges

B) Show that this series converges for all x values where x is a real number.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$a_{n+1} \le a_n$ when $n > N$ where N is an integer.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^{2n}}{(2n)!} = 0$$

$$\frac{x^{2(n+1)}}{(2(n+1))!} < \frac{x^{2n}}{(2n)!} \Rightarrow \frac{(2n)!}{(2n+2)!} < \frac{x^{2n}}{x^{2n+2}} \Rightarrow \frac{x^{2n+2}}{x^{2n}} < \frac{(2n+2)!}{(2n)!} \Rightarrow x^2 < (2n+2)(2n+1)$$

$$0 < (2n+2)(2n+1) - x^2 \Rightarrow 4n^2 + 6n + (2 - x^2) > 0$$

$4n^2 + 6n + (2 - x^2)$ is an open up parabola so when $n >$ the positive zero to the right

then $4n^2 + 6n + (2 - x^2) > 0. \Rightarrow$

$$N = \frac{-6 + \sqrt{6^2 - 4(4)(2 - x^2)}}{2(4)} = \frac{-6 + \sqrt{36 - 32 + 16x^2}}{2(4)} = \frac{-6 + \sqrt{4 + 16x^2}}{2(4)} = \frac{-3 + \sqrt{4x^2 + 1}}{4}$$

N can be found for any value of x.

C) Consider the function $f(x)$ where $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Determine if $f(x)$ has a relative minimum, relative maximum or neither at $x = 0$.
Give a reason for your answer.

$$f'(0) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{(0)^2}{2} + \frac{(0)^4}{4!} - \frac{(0)^6}{6!} + \cdots = 1 \Rightarrow \text{Neither}$$