AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: Unit 10*	**Convergence and Taylor Polynomials** **Free Response Question Review**	**Date:** April 13, 2020

* The Topics in this lesson will only be those that will be directly tested on the 2020 AP Calculus BC Exam

Topic Name	Topic #	Quick Synopsis
Working with **Geometric Series**	10.2	**CONVERGENCE OF A GEOMETRIC SERIES** **1.** If $\lvert r\rvert<1$, the geometric series $\sum_{n=0}^{\infty} ar^n$ converges **2.** If $\lvert r\rvert \ge 1$, the geometric series $\sum_{n=0}^{\infty} ar^n$ diverges. **SUM OF AN INFINITE GEOMETRIC SERIES** If $\lvert r\rvert<1$, the geometric series $\sum_{n=0}^{\infty} ar^n$ converges, and its sum is $S=\frac{a}{1-r}$, where a is the first term of the geometric series and r is the common ratio.
Harmonic and *p*-Series	10.5	**CONVERGENCE OF A *p*-SERIES** The p-series is defined by the following where p is a positive real number. $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$ **1.** converges if $p>1$, and **2.** diverges if $0<p\le 1$.
Alternating Series Test for Convergence	10.7	**ALTERNATING SERIES TEST** Let $a_n>0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converge if the following conditions are both met: 1. $\lim_{n\to\infty} a_n=0$ 2. $a_{n+1}\le a_n$ for all $n>N$ where N is an integer
Ratio Test for Convergence	10.8	**THE RATIO TEST** Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms. **1.** Let $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert=L$, a number. • If $L<1$, then the series $\sum_{n=1}^{\infty}a_n$ converges absolutely. • If $L=1$, then the ratio test provides no conclusive information about the convergence or divergence of $\sum_{n=1}^{\infty}a_n$. • If $L>1$, then the series $\sum_{n=1}^{\infty}a_n$ diverges. **2.** Let $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert\Rightarrow\infty$, then the series $\sum_{n=1}^{\infty}a_n$ diverges.
Finding **Taylor Polynomial Approximations** of Functions	10.11	**DEFINITIONS OF nTH TAYLOR POLYNOMIAL AND nTH MACLAURIN POLYNOMIAL** If f has n derivatives at c, then the polynomial $P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\cdots+\frac{f^{(n)}(c)(x-c)^n}{n!}$ is called the nth **Taylor polynomial** for f at c. If $c=0$: $P_n(x)=f(0)+f'(0)(x)+\frac{f''(0)(x)^2}{2!}+\cdots+\frac{f^{(n)}(0)(x)^n}{n!}$ is called the nth **Maclaurin polynomial** for f.

2020 FRQ Practice Problem BC1

BC1 Let $a_n = \frac{(-1)^n}{n^{p-2}}$ and $b_n = \frac{-2}{n^{6-p}}$

(a) Let $p = 2.5$. Show that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

$p = 2.5 \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$

Converges by Alternating Series Test,

1. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \to \frac{1}{\infty} \to 0$

2. $\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ because $\sqrt{n}$ is increasing

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{-2}{n^{7/2}}$

which is a convergent p-Series $p = \frac{7}{2} > 1$

(b) Find all integer values of p such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{p-2}}$ will converge if $p - 2 > 0 \Rightarrow p > 2$

$\sum_{n=1}^{\infty} \frac{-2}{n^{6-p}}$ will converge if $6 - p > 1 \Rightarrow p < 5$

$\therefore \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge when $p = 3$ or 4

(c) Let $p = 4$. Let $f(x)$ be a function with derivatives of all orders at $x = 2$ with $f(2) = -3$ and where $f^{(n)}(2) = n! \cdot a_n$ for $n \geq 1$. Find $P_3(x)$, the third degree Taylor polynomial for $f(x)$ centered at $x = 2$.

$f(2) = -3$

$f'(2) = 1! \cdot a_1 = \frac{(-1)^1}{1^{(4-2)}} = -1$

$f''(2) = 2! \cdot a_2 = 2 \cdot \frac{(-1)^2}{2^{(4-2)}} = \frac{1}{2}$

$f'''(2) = 3! \cdot a_3 = 6 \cdot \frac{(-1)^3}{3^{(4-2)}} = -\frac{6}{9} = -\frac{2}{3}$

$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$

$P_3(x) = -3 - 1(x-1) + \frac{1}{2 \cdot 2!}(x-2)^2 - \frac{2}{3 \cdot 3!}(x-2)^3$

$P_3(x) = -3 - (x-1) + \frac{1}{4}(x-2)^2 - \frac{1}{9}(x-2)^3$

(d) Using $P_3(x)$ that you found in part (c), find $P_3'(x)$. When $x = 3$, the series $\sum_{n=1}^{\infty} c_n$ is a p-series whose first three terms correspond to the three terms of $P_3'(x)$. Determine whether $\sum_{n=1}^{\infty} c_n$ converges or diverges when $x = 3$.

$$P_3(x) = -3 - (x-1) + \frac{1}{4}(x-2)^2 - \frac{1}{9}(x-2)^3$$

$$P_3'(x) = -1 + \frac{1}{2}(x-2) - \frac{1}{3}(x-2)^2$$

$$\sum_{n=1}^{\infty} c_n = -1 + \frac{1}{2}(x-2) - \frac{1}{3}(x-2)^2 + \cdots$$

At $x = 3$,

$$\sum_{n=1}^{\infty} c_n = -1 + \frac{1}{2}(3-2) - \frac{1}{3}(3-2)^2 + \cdots = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$$

The resulting series is the alternating harmonic series which converges.

2020 FRQ Practice Problem BC2

BC2 Consider the series $\sum_{n=0}^{\infty} a_n$ where $a_n = \dfrac{5(x+3)^n}{(-6)^n}$.

(a) Determine if $\sum_{n=0}^{\infty} a_n$ converges or diverges when $x = 1$.

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{5(x+3)^n}{(-6)^n}$$

At $x = 1$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{5(1+3)^n}{(-6)^n} = \sum_{n=0}^{\infty} \frac{5(4)^n}{(-6)^n} = \sum_{n=0}^{\infty} 5\left(-\frac{2}{3}\right)^n$

This is a geometric series where $r = -\frac{2}{3}$.

$|r| = \frac{2}{3} < 1 \Rightarrow$ the series converges.

(b) Let $\sum_{n=0}^{\infty} a_n = L$ where L is a real number. Show that there is a value of x such that $L = 15$.

$$\sum_{n=0}^{\infty} a_n = \frac{5}{1 - \left(\frac{x+3}{-6}\right)} = \frac{5}{1 + \frac{x+3}{6}} = \frac{5}{\frac{x+9}{6}} = \frac{30}{x+9} = 15$$

So, $15(x+9) = 30 \rightarrow x + 9 = 2 \rightarrow x = -7$

Where $x = -7$, $\sum_{n=0}^{\infty} \frac{5(-4)^n}{(-6)^n} = \sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$ which is a convergent geometric series.

(c) Let $d_n = \frac{a_n}{n+1}$. Find the interval of convergence for $\sum_{n=0}^{\infty} d_n$.

$$\sum_{n=0}^{\infty} d_n = \sum_{n=0}^{\infty} \frac{a_n}{n+1} = \sum_{n=0}^{\infty} \frac{5(x+3)^n}{(-6)^n(n+1)}$$

Using the ratio test, we obtain

$$\lim_{n\to\infty}\left|\frac{d_{n+1}}{d_n}\right| = \lim_{n\to\infty}\left|\frac{5(x+3)^{n+1}}{(-6)^{n+1}(n+2)} \cdot \frac{(-6)^n(n+1)}{5(x+3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+3)(n+1)}{(-6)^1(n+2)}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right) \cdot \lim_{n\to\infty}\left|\frac{(x+3)}{6}\right| = 1 \cdot \left|\frac{x+3}{6}\right|$$

To converge, $\left|\frac{x+3}{6}\right| < 1$.

$$\left|\frac{x+3}{6}\right| < 1 \to |x+3| < 6 \to -6 < x+3 < 6 \to -9 < x < 3$$

This series converges on $(-9,3]$

Check the endpoints.

$$x = -9 \Rightarrow \sum_{n=0}^{\infty} \frac{5(-9+3)^n}{(-6)^n(n+1)}$$

$= \sum_{n=0}^{\infty} \frac{5}{(n+1)}$ which is a divergent harmonic series

$x = 3$

$$\sum_{n=0}^{\infty} \frac{5(3+3)^n}{(-6)^n(n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{5(6)^n}{(-6)^n(n+1)}$$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{(n+1)}$ which converges by the AST.

(d) Let $f(x)$ be a function that is twice differentiable at all x values. If the first three terms of $\sum_{n=0}^{\infty} d_n$ are the second degree Taylor polynomial for $f(x)$ centered at $x = -3$, find $f''(-3)$.

$$\sum_{n=0}^{\infty} d_n = \frac{5(x+3)^0}{(-6)^0(0+1)} + \frac{5(x+3)^1}{(-6)^1(1+1)} + \frac{5(x+3)^2}{(-6)^2(2+1)} + \cdots$$

$$= 5 - \frac{5}{12}(x+3) + \frac{5}{108}(x+3)^2 + \cdots$$

$$\frac{f''(-3)}{2!} = \frac{5}{108} \Rightarrow f''(-3) = \frac{10}{108} = \frac{5}{54}$$

2020 FRQ Practice Problem BC3

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
1	-2	0	3	4
4	$-\frac{9}{4}$	$\frac{3}{2}$	$-\frac{9}{4}$	$\frac{9}{2}$

BC3 The functions f and g are differentiable for all orders at all x values. Selected values for f and several of its derivatives are given in the table above. The function g is defined by:

$$g(x) = 3x + \int_4^{4x} f(t)\,dt$$

(a) Find $P_3(x)$, the third degree Taylor polynomial for $f(x)$ centered at $x = 1$.

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$P_3(x) = -2 + (0)(x-1) + \frac{3}{2!}(x-1)^2 + \frac{4}{3!}(x-1)^3 = -2 + \frac{3}{2}(x-1)^2 + \frac{2}{3}(x-1)^3$$

(b) Find $T_3(x)$, the third degree Taylor polynomial for $g(x)$ centered at $x = 1$.

$$T_3(x) = g(1) + g'(1)(x-1) + \frac{g''(1)}{2!}(x-1)^2 + \frac{g'''(1)}{3!}(x-1)^3$$

$$T_3(x) = 3 - 6(x-1) + \frac{24}{2!}(x-1)^2 - \frac{144}{3!}(x-1)^3$$

$$= 3 - 6(x-1) + 12(x-1)^2 - 24(x-1)^3$$

$$g'(x) = 3 + 4\cdot f(4x)$$

$$g'(1) = 3 + 4\cdot f(4) = 3 + 4\left(-\frac{9}{4}\right) = -6$$

$$g''(x) = 16\cdot f'(4x)$$

$$g''(1) = 16\cdot f'(4) = 16\left(\frac{3}{2}\right) = 24$$

$$g'''(x) = 64\cdot f''(4x)$$

$$g'''(1) = 64\cdot f''(4) = 64\left(-\frac{9}{4}\right) = -144$$

(c) Let $\sum_{n=0}^{\infty} a_n$ be a geometric series whose first four terms are the four terms of $T_3(x)$ found in part (b).

Find $\sum_{n=0}^{\infty} a_n$ where $x = \frac{5}{4}$ or show that the series diverges.

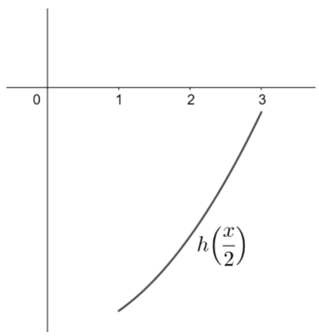
$$T_3(x) = 3 - 6(x-1) + 12(x-1)^2 - 24(x-1)^3$$

$$T_3\left(\frac{5}{4}\right) = 3 - 6\left(\frac{5}{4} - 1\right) + 12\left(\frac{5}{4} - 1\right)^2 - 24\left(\frac{5}{4} - 1\right)^3$$

$$\sum_{n=0}^{\infty} a_n = 3 - 6\left(\frac{1}{4}\right) + 12\left(\frac{1}{4}\right)^2 - 24\left(\frac{1}{4}\right)^3 + \cdots$$

$$= 3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$$

$$S = \frac{3}{1 - \left(-\frac{1}{2}\right)} = \frac{3}{3/2} = 2$$



(d) A portion of the function $h\left(\frac{x}{2}\right)$ is above. Explain why $h\left(\frac{x}{2}\right)$ could not be the graph of $f(x)$.

$f(1) = h\left(\frac{1}{2}\right)$ which we know nothing about

$f(4) = h(2)$ which is shown on the graph

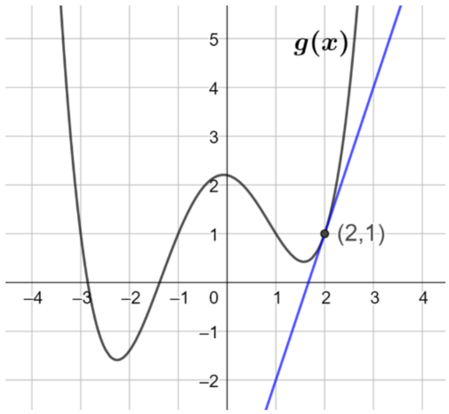
$f(4) = -\frac{9}{4} < 0$ $\qquad h(2) < 0$

$f'(4) = \frac{3}{2} > 0$ $\qquad h'(2) > 0$ because h is increasing

$f''(4) = -\frac{9}{4} < 0$ $\qquad h''(2) > 0$ because h is concave up

Based on the concavity, the graph of $h\left(\frac{x}{2}\right)$ could not be the graph of $f(x)$

2020 FRQ Practice Problem BC4



BC4 A function g has derivatives of all orders for all values of x. A portion of the graph of g is shown above with the line tangent to the graph of f at $x = 2$.

Let h be the function defined by $h(x) = x - 2 - \int_2^{2x} g(t)\,dt$.

(a) Find the second degree Taylor polynomial $T_2(x)$, for $h(x)$ centered at $x = 1$.

$$h(1) = ((1)-2) - \int_2^{2(1)} g(t)\,dt \quad = -1 - \int_2^2 g(t)\,dt \quad = -1$$

$$h'(x) = 1 - (g(2x)(2)) = 1 - 2g(2x)$$

$$h'(1) = 1 - 2g(2) = 1 - 2(1) = -1$$

$$h''(x) = -2g'(2x)(2) = -4g'(2x) \Rightarrow$$

$$h''(1) = -4g'(2) = -4(3) = -12$$

$$T_2(x) = h(1) + h'(1)(x-1) + \frac{h''(1)}{2!}(x-1)^2$$

$$T_2(x) = -1 + (-1)(x-1) + \frac{(-12)}{2!}(x-1)^2$$

$$= -1 - (x-1) - 6(x-1)^2$$

(b) Explain why $P_2(x) = 1 + 3(x-2) - \dfrac{5(x-2)^2}{2!}$ could not be the second degree Taylor polynomial for $g(x)$ centered at $x = 2$.

$g(2) = 1 \qquad g'(2) = 3$

$g(x)$ is concave up at $x = 2 \Rightarrow g''(2) > 0$

$P_2(x)$ 2nd degree term $= -\dfrac{5(x-2)^2}{2!} \Rightarrow g''(2) = -5$

so $P_2(x)$ can not be the second degree Taylor polynomial.

(c) Consider the geometric series $\sum_{n=0}^{\infty}\frac{a_n}{(2n)!}$ where the first three terms of a_n correspond to the three terms for $T_2(x)$. Find $\sum_{n=0}^{\infty}\frac{a_n}{(2n)!}$ when $x = 0$.

$$T_2(x) = -1-(x-1)-6(x-1)^2 = a_n$$

$$\sum_{n=0}^{\infty}\frac{a_n}{(2n)!} = \frac{-1}{0!}-\frac{1}{2!}(x-1)-\frac{6}{4!}(x-1)^2+\cdots = -1-\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\cdots$$

$$x=0 \Rightarrow \sum_{n=0}^{\infty}\frac{a_n}{(2n)!} = -1-\frac{1}{2}(-1)-\frac{1}{4}(-1)^2+\cdots = -1+\frac{1}{2}-\frac{1}{4}+\cdots$$

$$x=0 \Rightarrow \sum_{n=0}^{\infty}\frac{a_n}{(2n)!} = \frac{-1}{1-\left(-\frac{1}{2}\right)} = \frac{-2}{2+1} = -\frac{2}{3}$$