2019

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

AP 2020 Note:
Part d about the Alt. Series Error Board won't be tested

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Free Response Question 6

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- ☑ Student Samples
- ☑ Scoring Commentary

Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$\begin{aligned} &3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1) \\ &\quad = 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 \\ &\quad = 3 + x + x^2. \end{aligned}$$

$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$

(c)
$$\begin{aligned} h(1) &= \int_0^1 f(t)\,dt \\ &\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt \\ &= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1} \\ &= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48} \end{aligned}$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$

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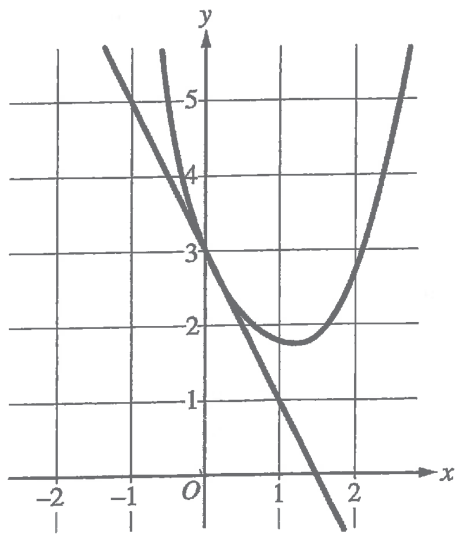

n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$\sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$$

$$3 - 2x + \frac{3x^2}{2!} - \frac{23x^3}{2 \cdot 3!}$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$1 + x + \frac{x^2}{2}$$

$$\left(1+x+\frac{x^2}{2}\right)\left(3 - 2x + \frac{3x^2}{2} - \frac{23x^3}{12}\right)$$

$$3 - 2x + \frac{3x^2}{2} + 3x - 2x^2 + \frac{3x^2}{2} + \ldots$$

$$3 + x + x^2\left(\frac{3}{2} - 2 + \frac{3}{2}\right)$$

$$3 + x + x^2$$

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(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$h(x) = 3x - x^2 + \frac{x^3}{2} - \frac{23x^4}{8 \cdot 3!}$

$h(1) = 3 - 1 + \frac{1}{2} - \frac{23}{8 \cdot 3!}$

$= 2 + \frac{1}{2} - \frac{23}{48}$

$= \frac{96 + 24 - 23}{48}$

$= \frac{97}{48}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$T_4(x) = \int_0^x \frac{54t^4}{4!}\,dt = \frac{54x^5}{5!}$

$T_4(1) = \frac{54}{5!} = \frac{54}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{54}{20 \cdot 6} = \frac{54}{120} = \frac{27}{60} = \frac{9}{20}$

error bound = fourth term = $\frac{9}{20}$ = 0.45

$\frac{9}{20} \leq 0.45$

error $\leq$ 0.45

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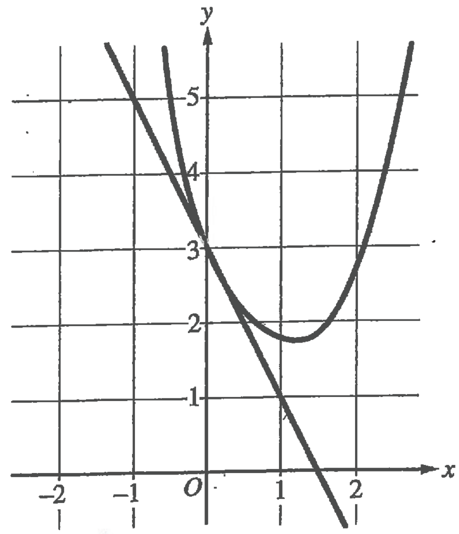

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6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$P_3 = \frac{3x^0}{0!} + \frac{-2x^1}{1!} + \frac{3x^2}{2!} - \frac{23x^3}{2\cdot 3!}$$

$$= 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$1 + x + \frac{x^2}{2}$$

$$\begin{array}{r} 3 - 2x + \frac{3}{2}x^2 \\ \times\ 1 + x + \frac{x^2}{2} \\ \hline 3 - 2x + \frac{3}{2}x^2 \\ 3x - 2x^2 \\ \frac{3}{2}x^2 \\ \hline \end{array}$$

$$\boxed{P_2 = 3 + x + x^2}$$

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NO CALCULATOR ALLOWED

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(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$h(x) = \int_0^x 3 - 2t + \frac{3}{2}t^2 - \frac{23}{36}t^3\,dt$

$= 3x - x^2 + \frac{1}{2}x^3 - \frac{23}{36}x^4$

$h(1) = 3 - 1 + \frac{1}{2} - \frac{23}{36}$

$= \frac{5}{2} - \frac{23}{36}$

$= \frac{67}{36}$

$\begin{array}{r} 4 \\ 18 \\ \times\ 5 \\ \hline 90 \\ -23 \\ \hline 67 \end{array}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$\frac{f^{(4)}(0)\,x^4}{4!} = \frac{54\,x^4}{4!} = \frac{27x^4}{12}$

$4! = 24 = 4 \cdot 3 \cdot 2$

$\frac{\frac{27}{5}x^5}{12} = \frac{27x^5}{60}$

$\frac{27}{60} \leq 0.45$

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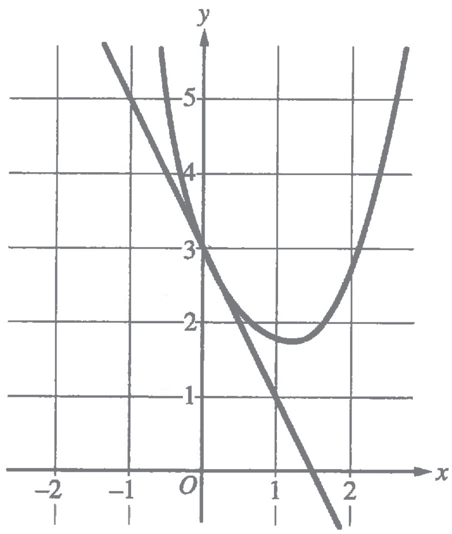

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6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!} + \frac{f'''(0)(x)^3}{3!} = T_3(x)$$

$$3 + \left(\frac{5-3}{-1-0}\right)x + \frac{3x^2}{2} + -\frac{23}{2}\cdot\frac{x^3}{6} = T_3(x)$$

$$\boxed{3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3 = T_3(x)}$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$M(x) = 1 + x + \frac{x^2}{2}$$

$$T_2(x) = f(0)\left(1 + x + \frac{x^2}{2}\right) = 3\left(1 + x + \frac{x^2}{2}\right)$$

$$\boxed{T_2(x) = 3 + 3x + \frac{3}{2}x^2}$$

Continue question 6 on page 21.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$h(1) = \int_0^1 f(t)\,dt$

$h(1) \approx 3 - 2(1) + \frac{3}{2}(1)^2 - \frac{23}{12}(1)^3$

$h(1) \approx 1 + \frac{3}{2} - \frac{23}{12}$

$h(1) \approx \frac{30}{12} - \frac{23}{12}$

$\boxed{h(1) \approx \frac{7}{12}}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$\frac{54x^4}{24} \leq 0.45$