The Higher Education Academy Engineering Subject Centre

Basic Engineering

Truth Tables and Boolean Algebra

F Hamer, R Horan & M Lavelle

The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of the fundamentals of Boolean Algebra through the use of truth tables.

 Email: chamer, rhoran, mlavelle@plymouth.ac.uk
Last Revision Date: May 18, 2005 Version 1.0

Table of Contents

The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Boolean Algebra (Introduction)

Boolean algebra is the algebra of propositions. Propositions will be denoted by upper case Roman letters, such as A or B, etc. Every proposition has two possible values: T when the proposition is *true* and F when the proposition is *false*.

The *negation* of A is written as $\neg A$ and read as "*not* A". If A is *true* then $\neg A$ is *false*. Conversely, if A is *false* then $\neg A$ is *true*. This relationship is displayed in the adjacent *truth table*.

A	$\neg A$
T	F
F	T

The second row of the table indicates that if A is *true* then $\neg A$ is *false*. The third row indicates that if A is *false* then $\neg A$ is *true*. Truth tables will be used throughout this package to verify that two propositions are logically equivalent. Two propositions are said to be ***logically equivalent*** if their truth tables have exactly the same values.

Example 1

Show that the propositions A and $\neg(\neg A)$ are logically equivalent.

Solution

From the definition of $\neg$ it follows that if $\neg A$ is *true* then $\neg(\neg A)$ is *false*, whilst if $\neg A$ is *false* then $\neg(\neg A)$ is *true*. This is encapsulated in the adjacent truth table.

A	$\neg A$	$\neg(\neg A)$
T	F	T
F	T	F

From the table it can be seen that when A takes the value *true*, the proposition $\neg(\neg A)$ also takes the value *true*, and when A takes the value *false*, the proposition $\neg(\neg A)$ also takes the value *false*. This shows that A and $\neg(\neg A)$ are *logically equivalent*, since their logical values are identical.

Logical equivalence may also be written as an equation which, in this case, is

$$A = \neg(\neg A)\,.$$

2. Conjunction ($\mathbf{A} \wedge \mathbf{B}$)

If A and B are two propositions then the *"conjunction"* of A and B, written as $A \wedge B$, and *read* as "*A and B*", is the proposition which is true ***if and only if*** both of A and B are true. The truth table for this is shown.

There are two possible values for each of the propositions A, B, so there are $2 \times 2 = 4$ possible assignments of values. This is seen in the truth table. Whenever truth tables are used, it is essential that every possible value is included.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table for $A \wedge B$

Example 2 Write out the truth table for the proposition $(A \wedge B) \wedge C$.

Solution The truth table will now contain $2 \times 2 \times 2 = 8$ rows, corresponding to the number of different possible values of the three propositions. It is shown on the next page.

In the adjacent table, the first three columns contain all possible values for A, B and C. The values in the column for $A \wedge B$ depend only upon the first two columns. The values of $(A \wedge B) \wedge C$ depend upon the values in the third and fourth columns.

A	B	C	$A \wedge B$	$(A \wedge B) \wedge C$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Truth table for $(A \wedge B) \wedge C$

EXERCISE 1. (Click on the green letters for the solutions.)

(a) Write out the truth table for $A \wedge (B \wedge C)$.

(b) Use **example 2** and part (a) to prove that

$$(A \wedge B) \wedge C = A \wedge (B \wedge C)\,.$$

3. Disjunction ($\mathbf{A} \vee \mathbf{B}$)

If A and B are two propositions then the *"disjunction"* of A and B, written as $A \vee B$, and *read* as "*A or B*", is the proposition which is true if either A or B, or both, are true. The truth table for this is shown.

As before, there are two possible values for each of the propositions A and B, so the number of possible assignments of values is $2 \times 2 = 4$. This is seen in the truth table.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Truth table for $A \vee B$

Example 3 Write out the truth table for the proposition $(A \vee B) \vee C$.

Solution The truth table will contain $2 \times 2 \times 2 = 8$ rows, corresponding to the number of different possible values of the three propositions. It is shown on the next page.

In the adjacent table, the first three columns contain all possible values for A, B and C. The values in the column for $A \vee B$ depend only upon the first two columns. The values of $(A \vee B) \vee C$ depend upon the values in the third and fourth columns.

A	B	C	$A \vee B$	$(A \vee B) \vee C$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

Truth table for $(A \vee B) \vee C$

EXERCISE 2. (Click on the green letters for the solutions.)

(a) Write out the truth table for $A \vee (B \vee C)$.

(b) Use **example 3** and part (a) to show that

$$(A \vee B) \vee C = A \vee (B \vee C)\,.$$

EXERCISE 3. Write out the truth tables for the following propositions. (Click on the green letters for the solutions.)
(a) $\neg(A \wedge B)$, (b) $(\neg A) \wedge B$, (c) $(\neg A) \wedge (\neg B)$,(d) $(\neg A) \vee (\neg B)$,

Quiz Using the results of exercise 3, which of the following is true?
(a) $(\neg A) \wedge B = (\neg A) \vee (\neg B)$, (b) $\neg(A \wedge B) = (\neg A) \vee (\neg B)$,
(c) $\neg(A \wedge B) = (\neg A) \wedge (\neg B)$, (d) $(\neg A) \wedge B = (\neg A) \wedge (\neg B)$.

EXERCISE 4. Write out the truth tables for the following propositions. (Click on the green letters for the solutions.)
(a) $A \wedge (B \vee C)$, (b) $(A \wedge B) \vee C$, (c) $(A \wedge B) \vee (A \wedge C)$.

Quiz Using the results of exercise 4, which of the following is true?
(a) $A \wedge (B \vee C) = (A \wedge B) \vee C$, (b) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$,
(c) $(A \wedge B) \vee C = (A \wedge B) \vee (A \wedge C)$,(d) None of these.

The three different operations $\neg$, $\wedge$, and $\vee$, (“*not*”, “*and*” and “*or*”), form the *algebraic* structure of Boolean algebra. What remains to be determined is the set of rules governing their interactions.

4. Rules of Boolean Algebra

(1a)	$A \wedge B$	$=$	$B \wedge A$
(1b)	$A \vee B$	$=$	$B \vee A$
(2a)	$A \wedge (B \wedge C)$	$=$	$(A \wedge B) \wedge C$
(2b)	$A \vee (B \vee C)$	$=$	$(A \vee B) \vee C$
(3a)	$A \wedge (B \vee C)$	$=$	$(A \wedge B) \vee (A \wedge C)$
(3b)	$A \vee (B \wedge C)$	$=$	$(A \vee B) \wedge (A \vee C)$
(4a)	$A \wedge A$	$=$	A
(4b)	$A \vee A$	$=$	A
(5a)	$A \wedge (A \vee B)$	$=$	A
(5b)	$A \vee (A \wedge B)$	$=$	A
(6a)	$A \wedge \neg A$	$=$	F
(6b)	$A \vee \neg A$	$=$	T
(7)	$\neg(\neg A)$	$=$	A
(8a)	$\neg(A \wedge B)$	$=$	$(\neg A) \vee (\neg B)$
(8a)	$\neg(A \vee B)$	$=$	$(\neg A) \wedge (\neg B)$

The following comments on these rules are useful:

All the rules can be verified by using truth tables.

Any rule labelled **(a)** can be obtained from the corresponding rule labelled **(b)** by replacing $\wedge$ with $\vee$ and $\vee$ with $\wedge$, and vice versa.

Rules **(1a)** and **(1b)** can easily be shown to be true.

Rules **(2a)** and **(2b)** were proved in exercises 1 and 2 respectively.

Rule **(3a)** was proved in exercise 3 and the quiz immediately following. Rules **(3a)** and **(3b)** are called the *distributive* rules.

Rule **(8a)** was proved in exercise 4 and the quiz immediately following. Rules **(8a)** and **(8b)** are called *De Morgan's laws*.

Rule **(2a)** means that $A \wedge B \wedge C$ have the same meaning given by $A \wedge (B \wedge C)$ or $(A \wedge B) \wedge C$. Similarly for $A \vee B \vee C$ from **(2b)**.

EXERCISE 5. Use truth tables to prove the following propositions. (Click on the green letters for the solutions.)

(a) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$,
(b) $A \wedge (A \vee B) = A$,
(c) $\neg(A \vee B) = \neg(A) \wedge \neg(B)$,
(d) $F \vee X = X$,
(e) $[A \wedge ((\neg A) \vee B)] \vee B = B$.

Truth tables are useful for proving that two expressions are equivalent but, often, the same result is easier to obtain using Boolean algebra.

Example 4 Simplify the expression $[A \wedge (\neg A \vee B)] \vee B$.

Solution
Using **(3a)**,

$$\begin{aligned} A \wedge (\neg A \vee B) = (A \wedge \neg A) \vee (A \wedge B) &= F \vee (A \wedge B), \text{ by } \textbf{(6a)}, \\ &= A \wedge B, \text{ by ex 5(d)}. \end{aligned}$$

Thus $[A \wedge (\neg A \vee B)] \vee B = (A \wedge B) \vee B = B$, by **(5b)**,

confirming the result of exercise 5(e), above.

5. Quiz on Boolean Algebra

Begin Quiz In each of the following, choose the simplified version of the given expression. (Use either truth tables, or Boolean algebra to simplify the expression.)

1. $(A \wedge \neg C) \vee (A \wedge B \wedge C) \vee (A \wedge C)$.
(a) T, (b) F, (c) $A \wedge B$, (d) A.

2. $[A \wedge B] \vee [A \wedge \neg B] \vee [(\neg A) \wedge B] \vee [(\neg A) \wedge (\neg B)]$.
(a) $A \wedge B$, (b) $A \vee B$, (c) T, (d) F.

3. $(A \wedge B \wedge C) \vee (\neg A) \vee (\neg B) \vee (\neg C)$.
(a) T, (b) F, (c) $A \wedge B$, (d) $A \wedge C$.

End Quiz

Score: Correct

Solutions to Exercises

Exercise 1(a)

The truth table for $A \wedge (B \wedge C)$ is constructed as follows:

In the first three columns, write all possible values for the propositions A, B and C, and use these to calculate the values in the fourth column. The final column is found by taking the *"conjunction"* of the first and fourth columns.

A	B	C	$B \wedge C$	$A \wedge (B \wedge C)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Click on the green square to return □

Exercise 1(b)
The logical equivalence of the propositions $(A \wedge B) \wedge C$ and $A \wedge (B \wedge C)$ can be seen by comparing their truth tables. The former is given in **example 2** and the latter in **exercise 1(a)**. These truth tables are identical, so

$$(A \wedge B) \wedge C = A \wedge (B \wedge C)\,.$$

Click on the green square to return □

Exercise 2(a)
The truth table for $A \vee (B \vee C)$ is shown below:

The first three columns contain all possible values for the propositions A, B and C. The fourth column represents the *"disjunction"* $(B \vee C)$ and the last column shows the *"disjunction"* of elements from the first and fourth columns.

A	B	C	$(B \vee C)$	$A \vee (B \vee C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

Click on the green square to return ☐

Exercise 2(b)

To prove the logical equivalence

$$(A \vee B) \vee C = A \vee (B \vee C)$$

compare the truth tables for both of these propositions. The former is given in **example 3** and the latter in **exercise 2(a)**. These truth tables are identical and therefore $(A \vee B) \vee C = A \vee (B \vee C)$.

Click on the green square to return □

Exercise 3(a)

The truth table for the proposition $\neg(A \wedge B)$ is given below.

The adjacent truth table contains $2\times 2 = 4$ rows corresponding to all possible values for the propositions A and B.

A	B	$A \wedge B$	$\neg(A \wedge B)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Click on the green square to return □

Exercise 3(b)

The truth table for the proposition $(\neg A) \wedge B$ is given below.

In the adjacent truth table the third column contains the *"negation"* of a given proposition A and the last column represents its *"conjunction"* with the proposition B.

A	B	$\neg A$	$(\neg A) \wedge B$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Click on the green square to return

Exercise 3(c)

The truth table for the proposition $(\neg A) \wedge (\neg B)$ is given below.

In the truth table the third and fourth columns contain the *"negation"* of the propositions A and B respectively, the last column is the *"conjunction"* of $\neg A$ and $\neg B$.

A	B	$\neg A$	$\neg B$	$(\neg A) \wedge (\neg B)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Click on the green square to return □

Exercise 3(d)

The truth table for the proposition $(\neg A) \vee (\neg B)$ is below.

In this truth table the third and fourth columns contain the *"negation"* of propositions A and B respectively, while the last column is the *"disjunction"* of $\neg A$ and $\neg B$.

A	B	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Click on the green square to return □

Exercise 4(a)

The truth table for the proposition $A \wedge (B \vee C)$ is below.

In the adjacent table, the first three columns contain all 8 possible values for propositions A, B and C. The fourth column shows the *"disjunction"* of B and C, while the last one is $A \wedge (B \vee C)$.

A	B	C	$B \vee C$	$A \wedge (B \vee C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Click on the green square to return ☐

Exercise 4(b)

The truth table for the proposition $(A \wedge B) \vee C$ is below.

The first three columns contain all possible values for propositions A, B and C. The fourth column shows the *"conjunction"* of A and B, while the last one is $(A \wedge B) \vee C$.

A	B	C	$A \wedge B$	$(A \wedge B) \vee C$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Click on the green square to return □

Exercise 4(c)
The truth table for the proposition $(A \wedge B) \vee (A \wedge C)$ is below.

The fourth and fifth columns show the *"conjunction"* of A and B, and A and C, respectively. The final column is the *"disjunction"* of these.

A	B	C	$A \wedge B$	$A \wedge C$	$(A \wedge B) \vee (A \wedge B)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Click on the green square to return □

Exercise 5(a) The truth tables for $A \vee (B \wedge C)$ and $(A \vee B) \wedge (A \vee C)$ are given below.

A	B	C	$A \vee (B \wedge C)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

A	B	C	$(A \vee B) \wedge (A \vee C)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Comparing them shows that

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)\,.$$

Click on the green square to return □

Exercise 5(b)

The truth table for $(A \vee B)$ is given below.

A	B	$A \vee B$	$A \wedge (A \vee B)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Comparing the first and the last columns above shows that

$$A \wedge (A \vee B) = A\,.$$

Click on the green square to return ☐

Exercise 5(c)
The truth table for $\neg(A \vee B)$ and $\neg(A) \wedge \neg(B)$ are given below.

A	B	$\neg(A \vee B)$
T	T	F
T	F	F
F	T	F
F	F	T

A	B	$(\neg A) \wedge (\neg B)$
T	T	F
T	F	F
F	T	F
F	F	T

The truth tables are identical so these propositions are logically equivalent, i.e.

$$\neg(A \vee B) = \neg(A) \wedge \neg(B) \,.$$

Click on the green square to return □

Exercise 5(d)

For an arbitrary proposition X, the *"disjunction"* of X and *"false"* (i.e. the proposition F) has the following truth table.

F	X	$F \vee X$
F	T	T
F	F	F

Therefore the relation

$$F \vee X = X$$

is correct.

Click on the green square to return □

Exercise 5(e)

For any two propositions A and B, construct the following truth table:

A	B	$(\neg A) \vee B$	$A \wedge ((\neg A) \vee B)$	$[A \wedge ((\neg A) \vee B)] \vee B$
T	T	T	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	F

Comparing the last column with the second one we find that

$$[A \wedge ((\neg A) \vee B)] \vee B = B\,.$$

Click on the green square to return □

Solutions to Quizzes

Solution to Quiz:

The truth tables from exercise 3(a) and exercise 3(d) are, respectively,

A	B	$\neg(A \wedge B)$
T	T	F
T	F	T
F	T	T
F	F	T

A	B	$(\neg A) \vee (\neg B)$
T	T	F
T	F	T
F	T	T
F	F	T

Since these are identical, it follows that

$$\neg(A \wedge B) = (\neg A) \vee (\neg B)\,.$$

End Quiz

Solution to Quiz:
The truth tables below are from exercise 4(a) and exercise 4(c), respectively.

A	B	C	$A \wedge (B \vee C)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

A	B	C	$(A \wedge B) \vee (A \wedge B)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

From these, it can be seen that

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)\,.$$

End Quiz